# Convex Optimization

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# Introduction

- mathematical optimization
- linear and convex optimization
- recent history

# Mathematical optimization

$$
\begin{array}{ll}
\text{minimize} & f_0(x_1, \ldots, x_n) \\
\text{subject to} & f_1(x_1, \ldots, x_n) \leq 0 \\
& \cdots \\
& f_m(x_1, \ldots, x_n) \leq 0
\end{array}
$$

- a mathematical model of a decision, design, or estimation problem
- generally intractable
- even simple looking nonlinear optimization problems can be very hard

# The famous exception: linear programming

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m \end{array}$$

- widely used since Dantzig introduced the simplex algorithm in 1948
- since 1950s, many applications in operations research, network optimization, finance, engineering, combinatorial optimization, . . .
- extensive theory (optimality conditions, sensitivity, . . . )
- there exist very efficient algorithms for solving linear programs

# Convex optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \end{array}$$

- objective and constraint functions are convex: for $0 \leq \theta \leq 1$

$$f_i(\theta x + (1-\theta)y) \leq \theta f_i(x) + (1-\theta) f_i(y)$$

- can be solved globally, with similar (polynomial-time) complexity as LPs
- surprisingly many problems can be solved via convex optimization
- provides tractable heuristics and relaxations for non-convex problems

# History

- 1940s: linear programming

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m \end{array}$$

- 1950s: quadratic programming

- 1960s: geometric programming

- 1990s: semidefinite programming, second-order cone programming, quadratically constrained quadratic programming, robust optimization, sum-of-squares programming, . . .

# New applications since 1990

- linear matrix inequality techniques in control
- support vector machine training via quadratic programming
- semidefinite programming relaxations in combinatorial optimization
- circuit design via geometric programming
- $\ell_1$-norm optimization for sparse signal reconstruction
- applications in structural optimization, statistics, signal processing, communications, image processing, computer vision, quantum information theory, finance, power distribution, . . .

# Advances in convex optimization algorithms

**interior-point methods**

- 1984 (Karmarkar): first practical polynomial-time algorithm for LP
- 1984-1990: efficient implementations for large-scale LPs
- around 1990 (Nesterov & Nemirovski): polynomial-time interior-point methods for nonlinear convex programming
- since 1990: extensions and high-quality software packages

**fast first-order algorithms**

- similar to gradient descent, but with better convergence properties
- based on Nesterov's optimal-rate gradient methods from 1980s
- extend to certain nondifferentiable or constrained problems

# Overview

1. Basic theory and convex modeling
   - convex sets and functions
   - common problem classes and applications

2. Interior-point methods for conic optimization
   - conic optimization
   - barrier methods
   - symmetric primal-dual methods

3. First-order methods
   - gradient algorithms
   - dual techniques

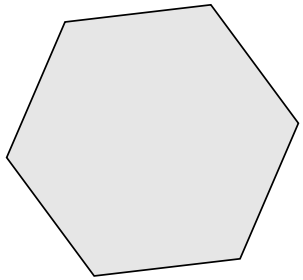
# Convex sets and functions

- convex sets
- convex functions
- operations that preserve convexity

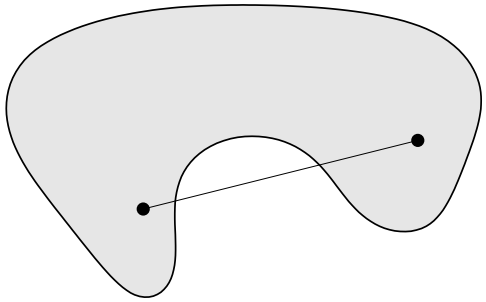
# Convex set

contains the line segment between any two points in the set

$$x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \Longrightarrow \quad \theta x_1 + (1-\theta)x_2 \in C$$



convex



not convex

not convex

# Basic examples

**affine set:** solution set of linear equations $Ax = b$

**halfspace:** solution of one linear inequality $a^T x \leq b$ $(a \neq 0)$

**polyhedron:** solution of finitely many linear inequalities $Ax \leq b$

**ellipsoid:** solution of quadratic inquality

$$(x - x_{\mathrm{c}})^T A(x - x_{\mathrm{c}}) \leq 1 \qquad (A \text{ positive definite})$$

**norm ball:** solution of $\|x\| \leq R$ (for any norm)

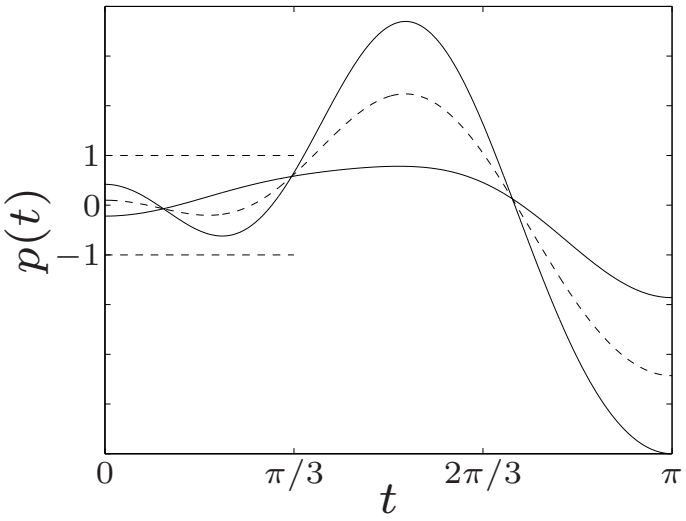
**positive semidefinite cone:** $\mathbf{S}_+^n = \{X \in \mathbf{S}^n \mid X \succeq 0\}$

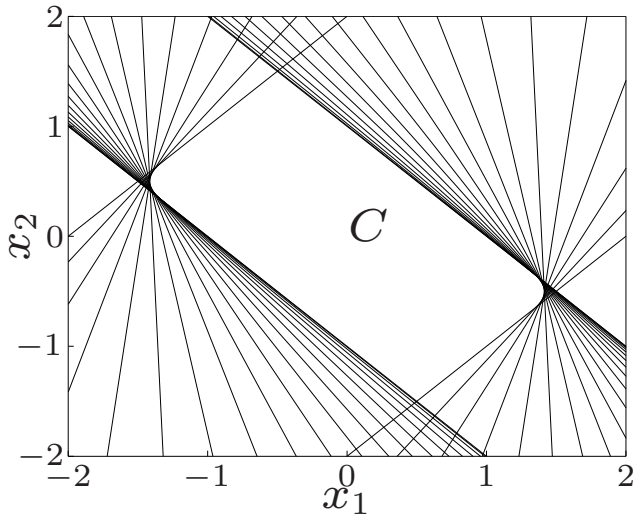
the **intersection** of any number of convex sets is convex

# Example of intersection property

$$C = \{x \in \mathbf{R}^n \mid |p(t)| \leq 1 \text{ for } |t| \leq \pi/3\}$$

where $p(t) = x_1 \cos t + x_2 \cos 2t + \cdots + x_n \cos nt$




$C$ is intersection of infinitely many halfspaces, hence convex

# Outline

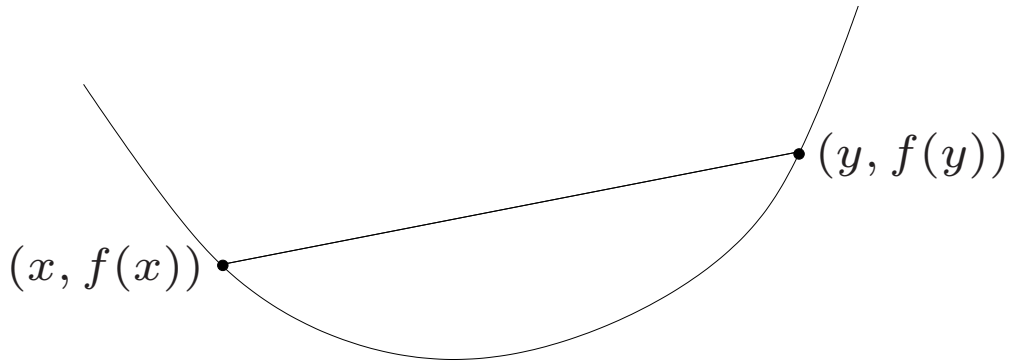
- convex sets
- **convex functions**
- operations that preserve convexity

# Convex function

domain $\mathbf{dom}\, f$ is a convex set and Jensen's inequality holds:

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$$

for all $x, y \in \mathbf{dom}\, f$, $0 \leq \theta \leq 1$




$f$ is concave if $-f$ is convex

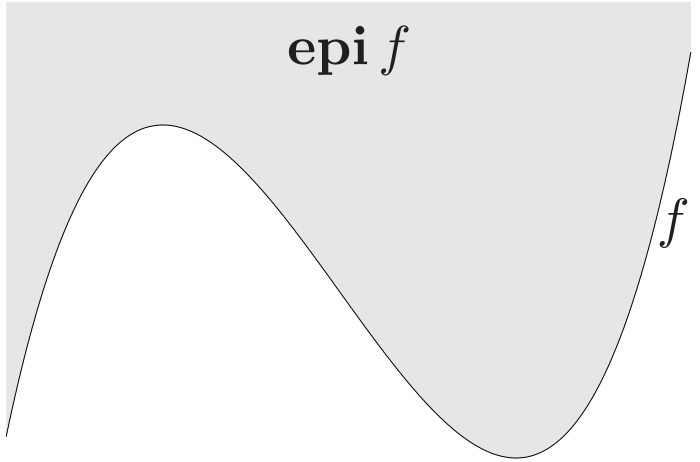
# Examples

- linear and affine functions are convex and concave
- $\exp x$, $-\log x$, $x \log x$ are convex
- $x^\alpha$ is convex for $x > 0$ and $\alpha \geq 1$ or $\alpha \leq 0$; $|x|^\alpha$ is convex for $\alpha \geq 1$
- norms are convex
- quadratic-over-linear function $x^T x/t$ is convex in $x$, $t$ for $t > 0$
- geometric mean $(x_1 x_2 \cdots x_n)^{1/n}$ is concave for $x \succeq 0$
- $\log \det X$ is concave on set of positive definite matrices
- $\log(e^{x_1} + \cdots e^{x_n})$ is convex

# Epigraph and sublevel set

**epigraph**: $\mathbf{epi}\, f = \{(x,t) \mid x \in \mathbf{dom}\, f,\ f(x) \le t\}$

a function is convex if and only its epigraph is a convex set



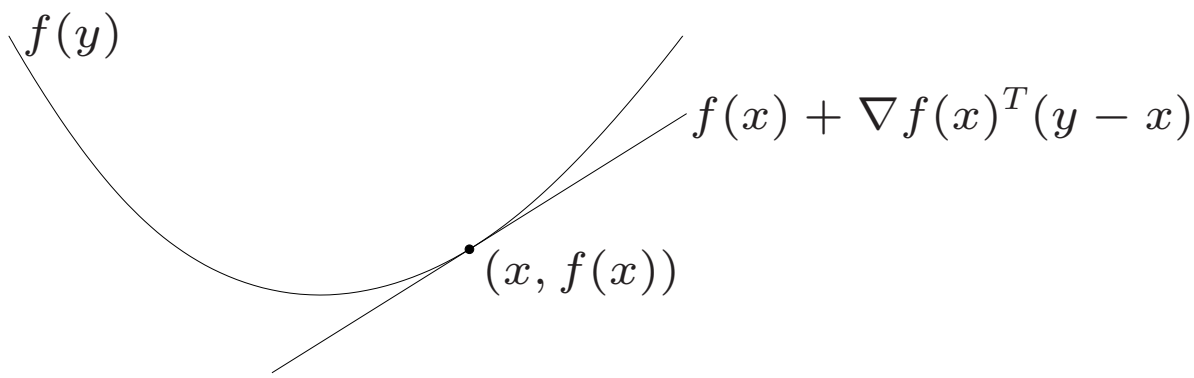

**sublevel sets**: $C_\alpha = \{x \in \mathbf{dom}\, f \mid f(x) \le \alpha\}$

the sublevel sets of a convex function are convex (converse is false)

# Differentiable convex functions

differentiable $f$ is convex if and only if $\mathbf{dom}\, f$ is convex and

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) \quad \text{for all } x, y \in \mathbf{dom}\, f$$




twice differentiable $f$ is convex if and only if $\mathbf{dom}\, f$ is convex and

$$\nabla^2 f(x) \succeq 0 \quad \text{for all } x \in \mathbf{dom}\, f$$

# Outline

- convex sets
- convex functions
- **operations that preserve convexity**

# Methods for establishing convexity of a function

1. verify definition

2. for twice differentiable functions, show $\nabla^2 f(x) \succeq 0$

3. show that $f$ is obtained from simple convex functions by operations that preserve convexity

   - nonnegative weighted sum
   - composition with affine function
   - pointwise maximum and supremum
   - minimization
   - composition
   - perspective

# Positive weighted sum & composition with affine function

**nonnegative multiple:** $\alpha f$ is convex if $f$ is convex, $\alpha \geq 0$

**sum:** $f_1 + f_2$ convex if $f_1, f_2$ convex (extends to infinite sums, integrals)

**composition with affine function:** $f(Ax + b)$ is convex if $f$ is convex

**examples**

- logarithmic barrier for linear inequalities

$$f(x) = -\sum_{i=1}^{m} \log(b_i - a_i^T x)$$

- (any) norm of affine function: $f(x) = \|Ax + b\|$

# Pointwise maximum

$$f(x) = \max\{f_1(x), \ldots, f_m(x)\}$$

is convex if $f_1, \ldots, f_m$ are convex

**example:** sum of $r$ largest components of $x \in \mathbf{R}^n$

$$f(x) = x_{[1]} + x_{[2]} + \cdots + x_{[r]}$$

is convex ($x_{[i]}$ is $i$th largest component of $x$)

proof:

$$f(x) = \max\{x_{i_1} + x_{i_2} + \cdots + x_{i_r} \mid 1 \leq i_1 < i_2 < \cdots < i_r \leq n\}$$

# Pointwise supremum

$$g(x) = \sup_{y \in \mathcal{A}} f(x, y)$$

is convex if $f(x, y)$ is convex in $x$ for each $y \in \mathcal{A}$

**example:** maximum eigenvalue of symmetric matrix

$$\lambda_{\max}(X) = \sup_{\|y\|_2 = 1} y^T X y$$

# Minimization

$$h(x) = \inf_{y \in C} f(x, y)$$

is convex if $f(x, y)$ is convex in $x, y$ and $C$ is a convex set

**examples**

- distance to a convex set $C$: $h(x) = \inf_{y \in C} \|x - y\|$
- optimal value of linear program as function of righthand side

$$h(x) = \inf_{y: Ay \leq x} c^T y$$

follows by taking

$$f(x, y) = c^T y, \qquad \mathbf{dom}\, f = \{(x, y) \mid Ay \leq x\}$$

# Composition

composition of $g : \mathbf{R}^n \to \mathbf{R}$ and $h : \mathbf{R} \to \mathbf{R}$:

$$f(x) = h(g(x))$$

$f$ is convex if

$g$ convex, $h$ convex and nondecreasing
$g$ concave, $h$ convex and nonincreasing

(if we assign $h(x) = \infty$ for $x \in \mathbf{dom}\, h$)

**examples**

- $\exp g(x)$ is convex if $g$ is convex
- $1/g(x)$ is convex if $g$ is concave and positive

# Perspective

the **perspective** of a function $f : \mathbf{R}^n \to \mathbf{R}$ is the function $g : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$,

$$g(x,t) = tf(x/t)$$

$g$ is convex if $f$ is convex on $\mathbf{dom}\, g = \{(x,t) \mid x/t \in \mathbf{dom}\, f,\ t > 0\}$

**examples**

- perspective of $f(x) = x^T x$ is quadratic-over-linear function

$$g(x,t) = \frac{x^T x}{t}$$

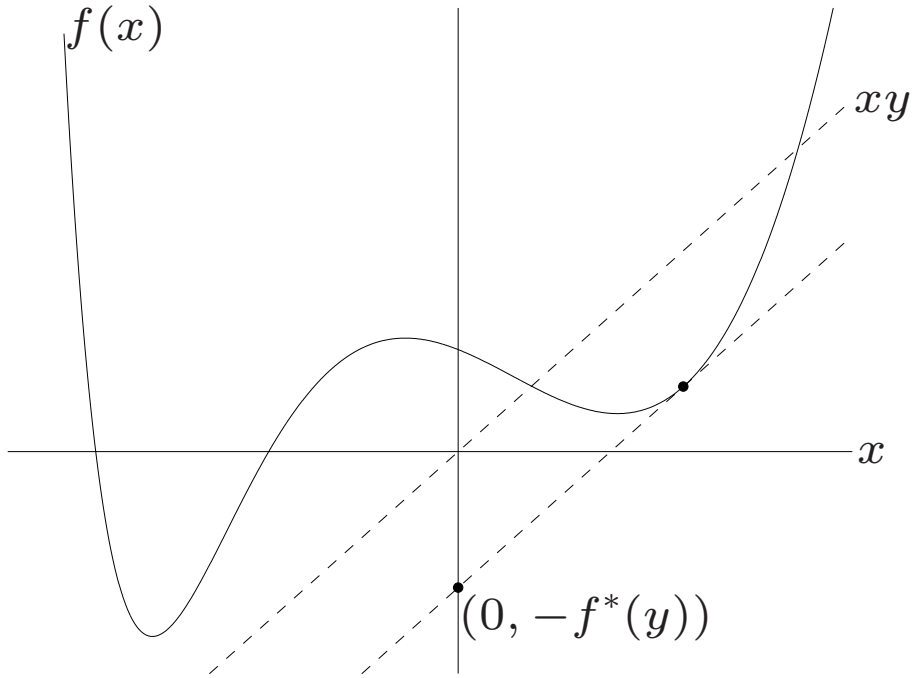
- perspective of negative logarithm $f(x) = -\log x$ is relative entropy

$$g(x,t) = t\log t - t\log x$$

# Conjugate function

the **conjugate** of a function $f$ is

$$f^*(y) = \sup_{x \in \mathbf{dom}\, f} (y^T x - f(x))$$




$f^*$ is convex (even if $f$ is not)

# Examples

**convex quadratic function** ($Q \succ 0$)

$$f(x) = \frac{1}{2}x^T Q x \qquad\qquad f^*(y) = \frac{1}{2}y^T Q^{-1} y$$

**negative entropy**

$$f(x) = \sum_{i=1}^n x_i \log x_i \qquad\qquad f^*(y) = \sum_{i=1}^n e^{y_i} - 1$$

**norm**

$$f(x) = \|x\| \qquad\qquad f^*(y) = \begin{cases} 0 & \|y\|_* \leq 1 \\ +\infty & \text{otherwise} \end{cases}$$

**indicator function** ($C$ convex)

$$f(x) = I_C(x) = \begin{cases} 0 & x \in C \\ +\infty & \text{otherwise} \end{cases} \qquad\qquad f^*(y) = \sup_{x \in C} y^T x$$

# Convex optimization problems

- linear programming
- quadratic programming
- geometric programming
- second-order cone programming
- semidefinite programming
- modeling software

# Convex optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

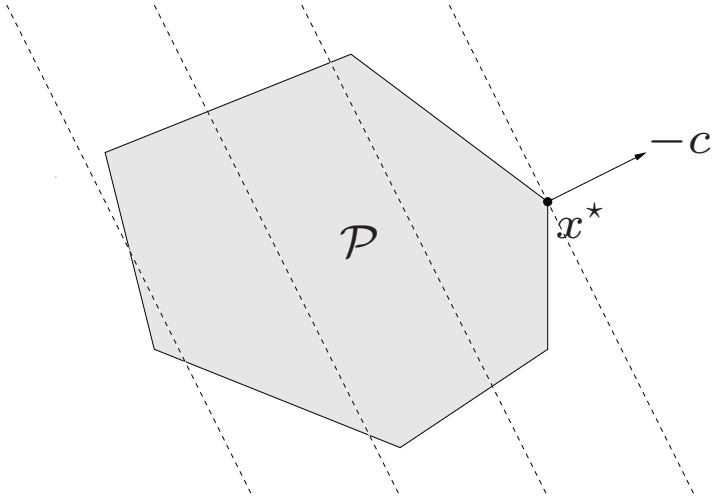
$f_0,\ f_1,\ \ldots,\ f_m$ are convex functions

- feasible set is convex
- locally optimal points are globally optimal
- tractable, in theory and practice

# Linear program (LP)

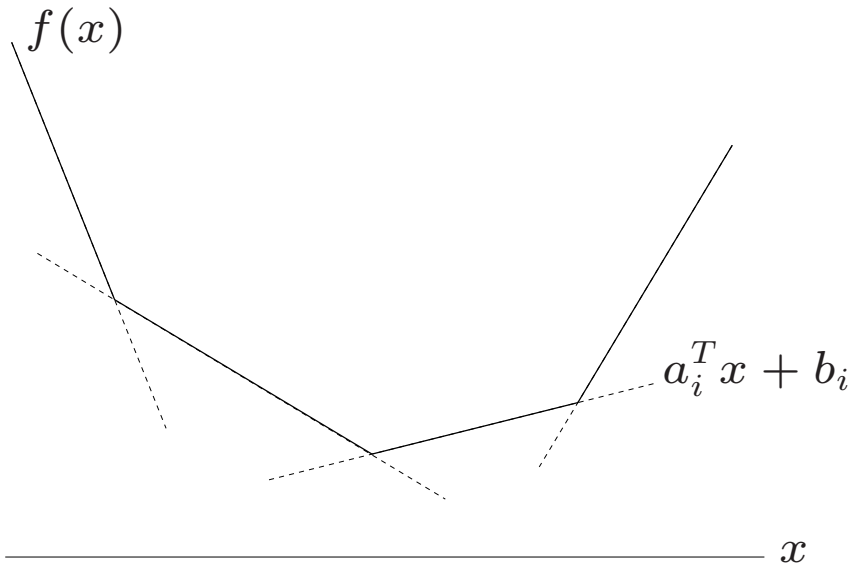
$$\begin{array}{ll} \text{minimize} & c^T x + d \\ \text{subject to} & Gx \leq h \\ & Ax = b \end{array}$$

- inequality is componentwise vector inequality
- convex problem with affine objective and constraint functions
- feasible set is a polyhedron



# Piecewise-linear minimization

$$\text{minimize} \quad f(x) = \max_{i=1,\ldots,m} (a_i^T x + b_i)$$




**equivalent linear program**

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & a_i^T x + b_i \leq t, \quad i = 1, \ldots, m \end{array}$$

an LP with variables $x$, $t \in \mathbf{R}$

# $\ell_1$-Norm and $\ell_\infty$-norm minimization

**$\ell_1$-norm approximation and equivalent LP** ($\|y\|_1 = \sum_k |y_k|$)

$$\begin{array}{ll} \text{minimize} & \|Ax-b\|_1 \end{array} \qquad \begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{n} y_i \\ \text{subject to} & -y \leq Ax-b \leq y \end{array}$$

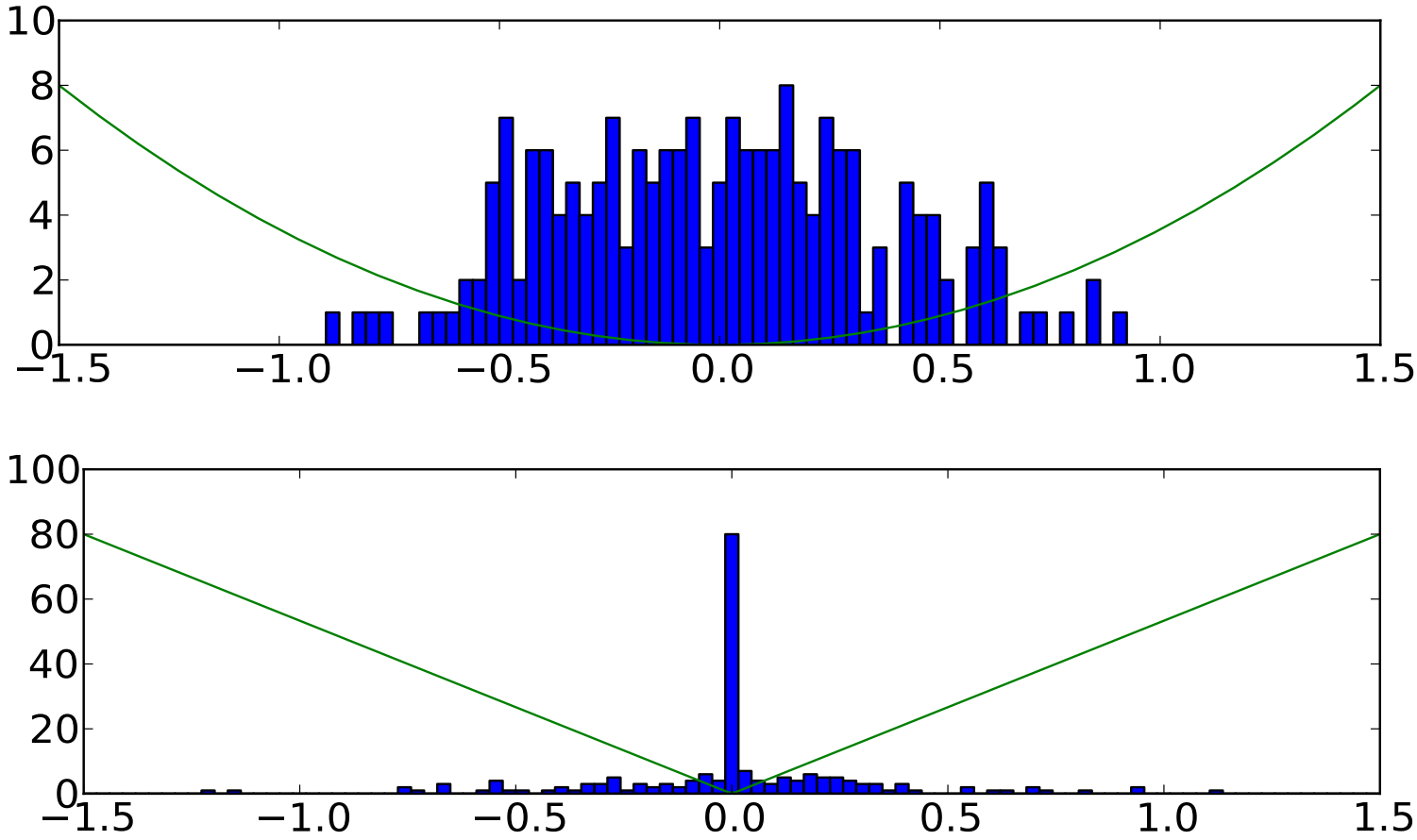
**$\ell_\infty$-norm approximation** ($\|y\|_\infty = \max_k |y_k|$)

$$\begin{array}{ll} \text{minimize} & \|Ax-b\|_\infty \end{array} \qquad \begin{array}{ll} \text{minimize} & y \\ \text{subject to} & -y\mathbf{1} \leq Ax-b \leq y\mathbf{1} \end{array}$$

($\mathbf{1}$ is vector of ones)

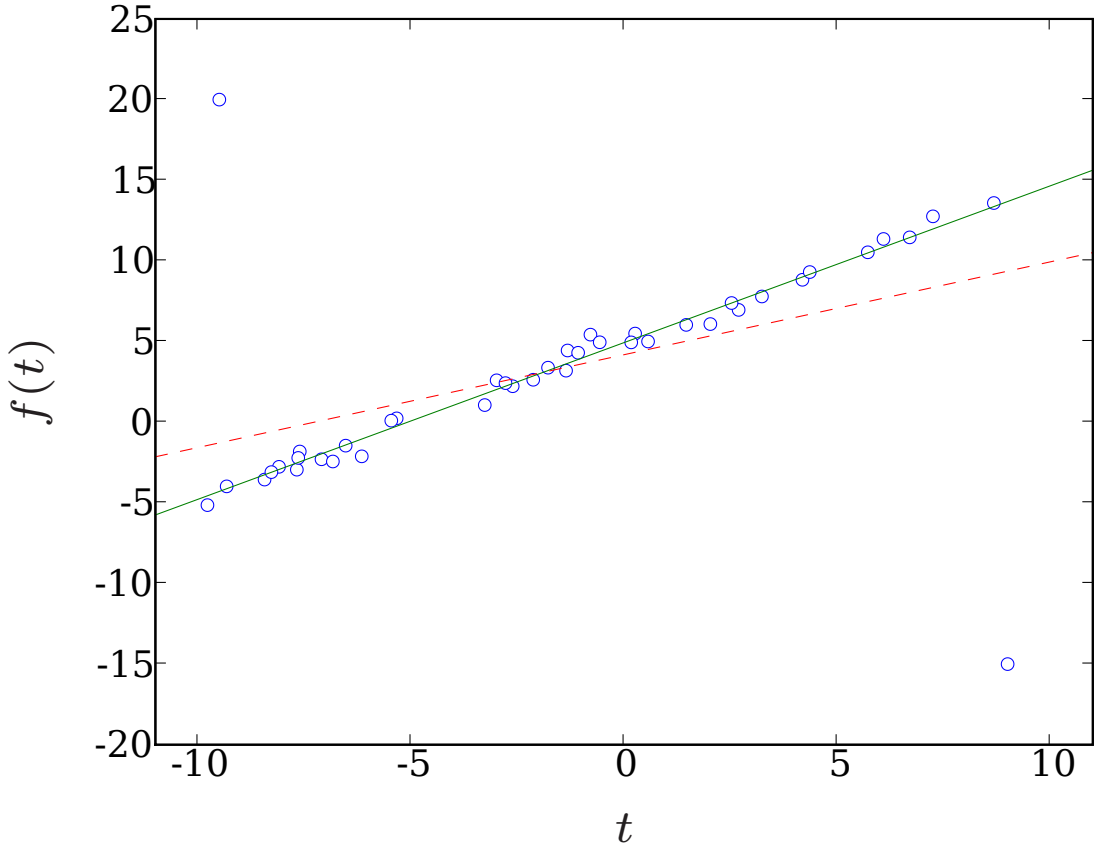
**example:** histograms of residuals (with $A$ is $100 \times 30$)

minimize $\|Ax - b\|_2$ minimize $\|Ax - b\|_1$




1-norm distribution is wider with a high peak at zero

# Robust regression

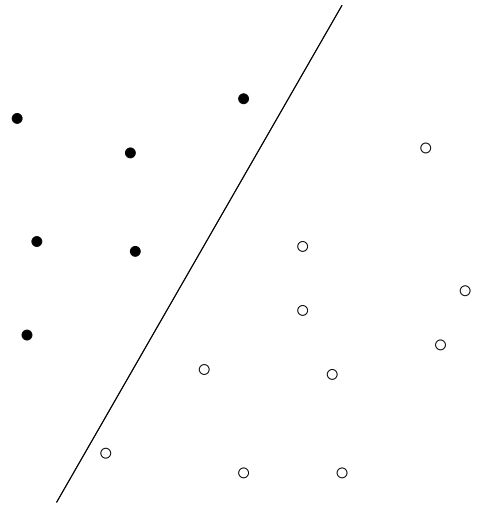



- 42 points $t_i$, $y_i$ (circles), including two outliers
- function $f(t) = \alpha + \beta t$ fitted using 2-norm (dashed) and 1-norm

# Linear discrimination

separate two sets of points $\{x_1, \ldots, x_N\}$, $\{y_1, \ldots, y_M\}$ by a hyperplane

$$a^T x_i + b > 0, \quad i = 1, \ldots, N$$
$$a^T y_i + b < 0, \quad i = 1, \ldots, M$$

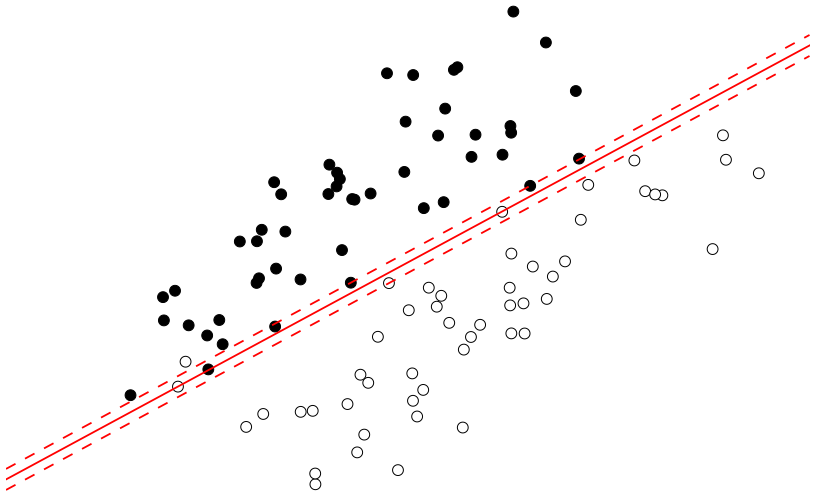


homogeneous in $a$, $b$, hence equivalent to the linear inequalities (in $a$, $b$)

$$a^T x_i + b \geq 1, \quad i = 1, \ldots, N, \qquad a^T y_i + b \leq -1, \quad i = 1, \ldots, M$$

# Approximate linear separation of non-separable sets

$$\text{minimize} \quad \sum_{i=1}^{N} \max\{0, 1 - a^T x_i - b\} + \sum_{i=1}^{M} \max\{0, 1 + a^T y_i + b\}$$

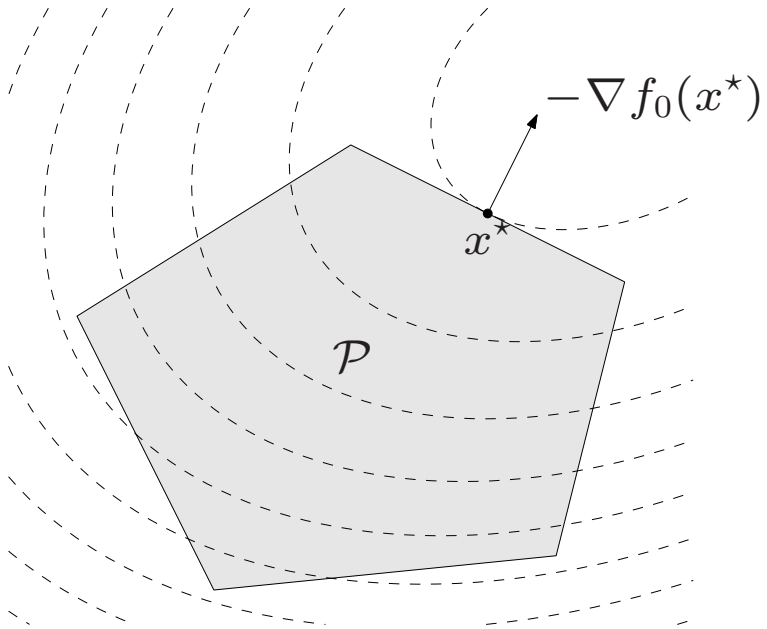


- a piecewise-linear minimization problem in $a$, $b$; equivalent to an LP
- can be interpreted as a heuristic for minimizing #misclassified points

# Quadratic program (QP)

$$\begin{array}{ll} \text{minimize} & (1/2)x^T Px + q^T x + r \\ \text{subject to} & Gx \leq h \end{array}$$

- $P \in \mathbf{S}_+^n$, so objective is convex quadratic
- minimize a convex quadratic function over a polyhedron



# Linear program with random cost

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx \leq h \end{array}$$

- $c$ is random vector with mean $\bar{c}$ and covariance $\Sigma$
- hence, $c^T x$ is random variable with mean $\bar{c}^T x$ and variance $x^T \Sigma x$

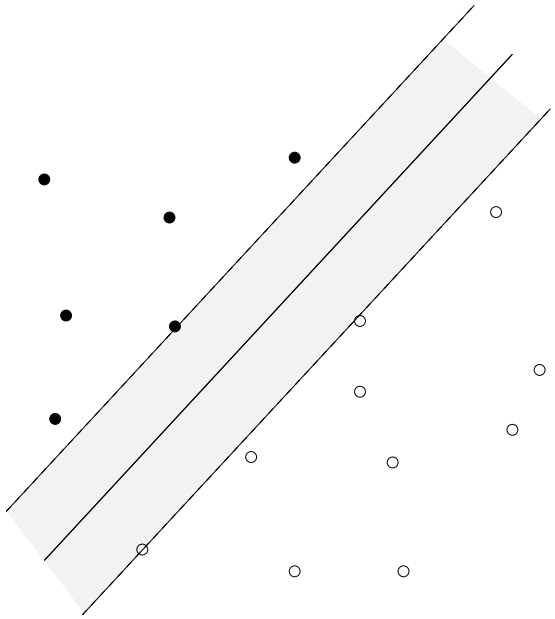
**expected cost-variance trade-off**

$$\begin{array}{ll} \text{minimize} & \mathbf{E}\, c^T x + \gamma\, \mathbf{var}(c^T x) = \bar{c}^T x + \gamma x^T \Sigma x \\ \text{subject to} & Gx \leq h \end{array}$$

$\gamma > 0$ is risk aversion parameter

# Robust linear discrimination

$$\begin{aligned}\mathcal{H}_1 &= \{z \mid a^T z + b = 1\} \\ \mathcal{H}_2 &= \{z \mid a^T z + b = -1\}\end{aligned}$$



distance between hyperplanes is $2/\|a\|_2$

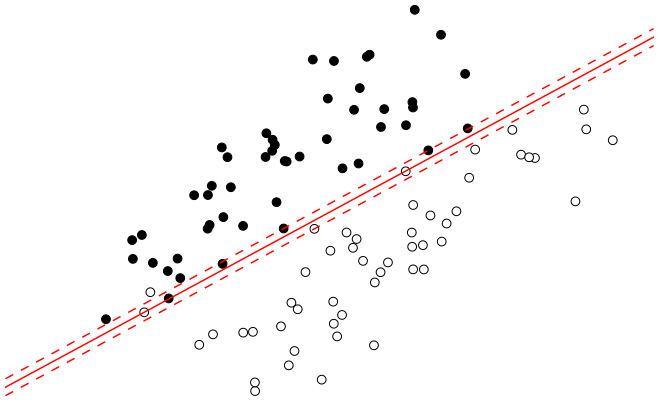
to separate two sets of points by maximum margin,

$$\begin{array}{ll}\text{minimize} & \|a\|_2^2 = a^T a \\ \text{subject to} & a^T x_i + b \geq 1, \quad i = 1, \ldots, N \\ & a^T y_i + b \leq -1, \quad i = 1, \ldots, M\end{array}$$

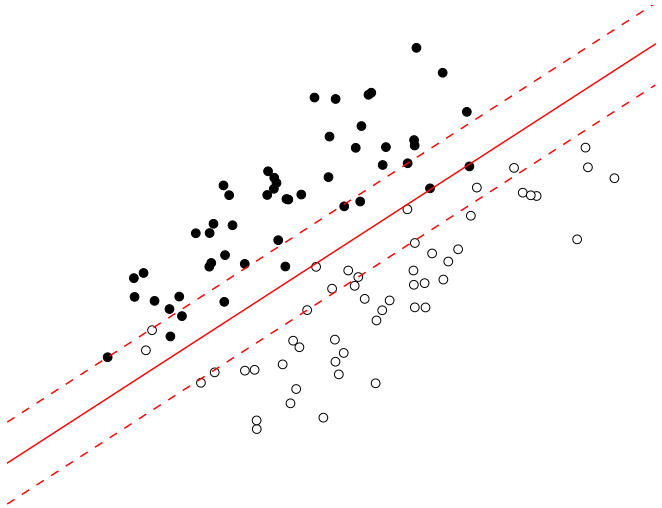
a quadratic program in $a$, $b$

# Support vector classifier

$$\text{min.} \quad \gamma\|a\|_2^2 + \sum_{i=1}^{N} \max\{0, 1 - a^T x_i - b\} + \sum_{i=1}^{M} \max\{0, 1 + a^T y_i + b\}$$



$\gamma = 0$



$\gamma = 10$

equivalent to a QP

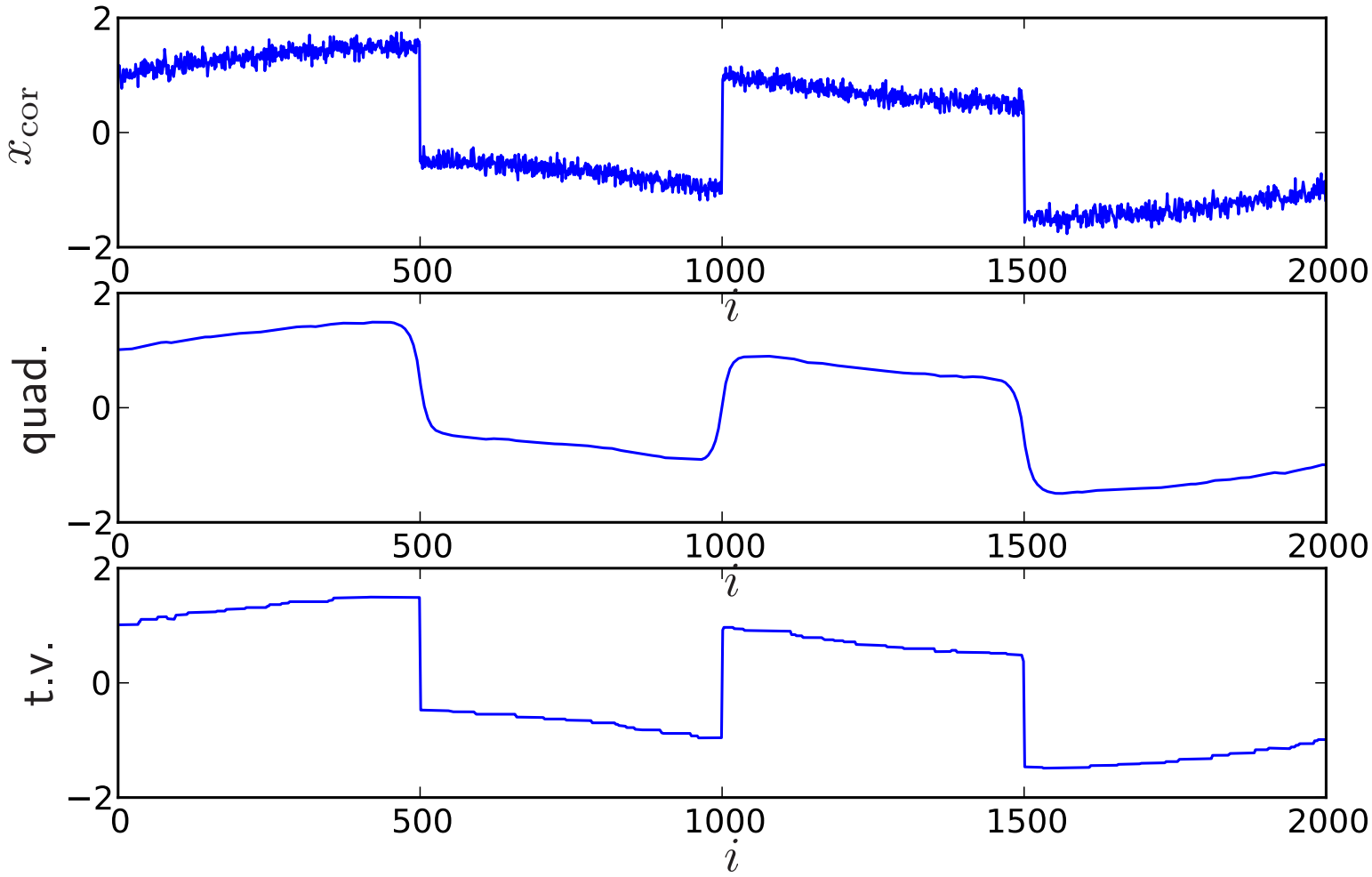
# Total variation signal reconstruction

$$\text{minimize} \quad \|\hat{x} - x_{\text{cor}}\|_2 + \gamma\phi(\hat{x})$$

- $x_{\text{cor}} = x + v$ is corrupted version of unknown signal $x$, with noise $v$
- variable $\hat{x}$ (reconstructed signal) is estimate of $x$
- $\phi : \mathbf{R}^n \to \mathbf{R}$ is quadratic or total variation smoothing penalty

$$\phi_{\text{quad}}(\hat{x}) = \sum_{i=1}^{n-1} (\hat{x}_{i+1} - \hat{x}_i)^2, \qquad \phi_{\text{tv}}(\hat{x}) = \sum_{i=1}^{n-1} |\hat{x}_{i+1} - \hat{x}_i|$$

**example:** $x_{\rm cor}$, and reconstruction with quadratic and t.v. smoothing




- quadratic smoothing smooths out noise and sharp transitions in signal
- total variation smoothing preserves sharp transitions in signal

# Geometric programming

**posynomial function**

$$f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}, \qquad \mathbf{dom}\, f = \mathbf{R}^n_{++}$$

with $c_k > 0$

**geometric program (GP)**

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 1, \quad i = 1, \ldots, m \end{array}$$

with $f_i$ posynomial

# Geometric program in convex form

change variables to

$$y_i = \log x_i,$$

and take logarithm of cost, constraints

**geometric program** in convex form:

$$\begin{array}{ll} \text{minimize} & \log\left(\displaystyle\sum_{k=1}^{K} \exp(a_{0k}^T y + b_{0k})\right) \\ \text{subject to} & \log\left(\displaystyle\sum_{k=1}^{K} \exp(a_{ik}^T y + b_{ik})\right) \leq 0, \quad i = 1, \ldots, m \end{array}$$

$b_{ik} = \log c_{ik}$

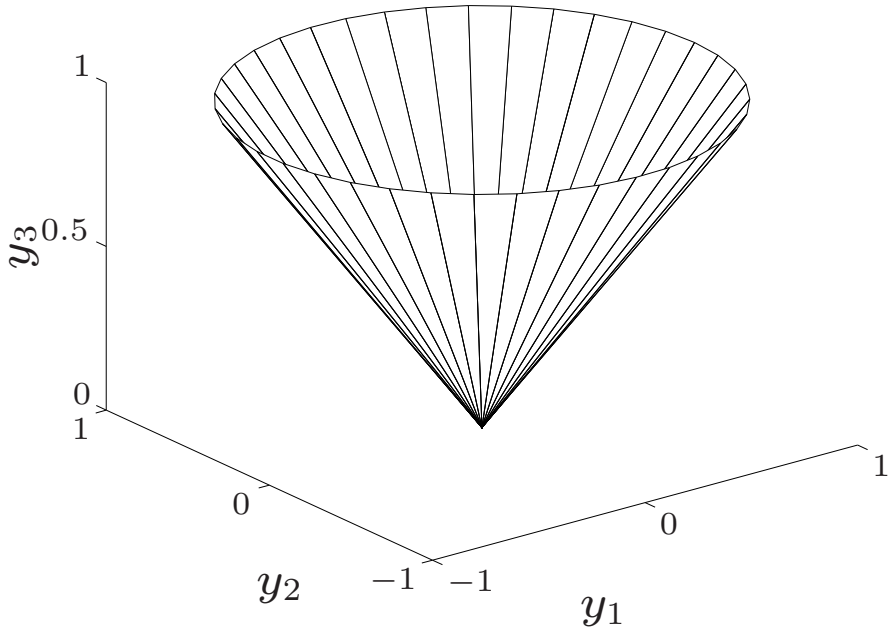
# Second-order cone program (SOCP)

$$\begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, \ldots, m \end{array}$$

- $\|\cdot\|_2$ is Euclidean norm $\|y\|_2 = \sqrt{y_1^2 + \cdots + y_n^2}$
- constraints are nonlinear, nondifferentiable, convex

constraints are inequalities
w.r.t. second-order cone:

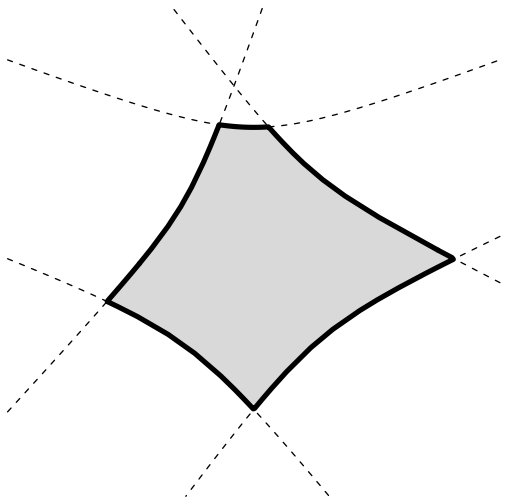
$$\left\{ y \,\middle|\, \sqrt{y_1^2 + \cdots + y_{p-1}^2} \leq y_p \right\}$$



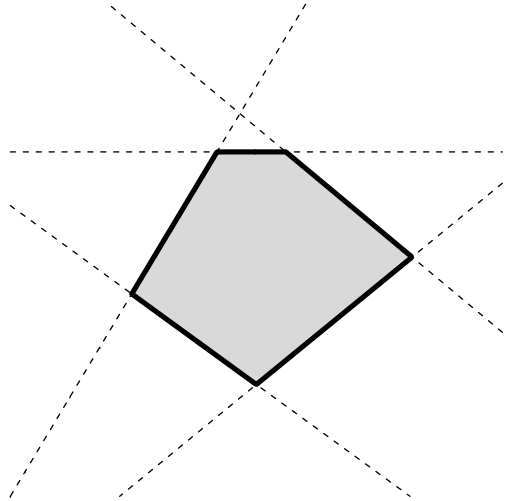
# Robust linear program (stochastic)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{prob}(a_i^T x \leq b_i) \geq \eta, \quad i = 1, \ldots, m \end{array}$$

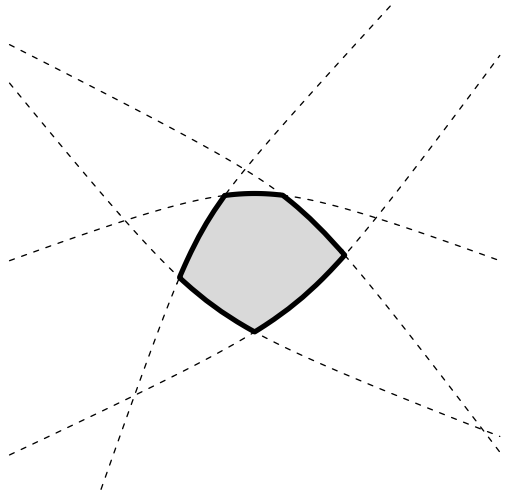
- $a_i$ random and normally distributed with mean $\bar{a}_i$, covariance $\Sigma_i$
- we require that $x$ satisfies each constraint with probability exceeding $\eta$



$\eta = 10\%$



$\eta = 50\%$



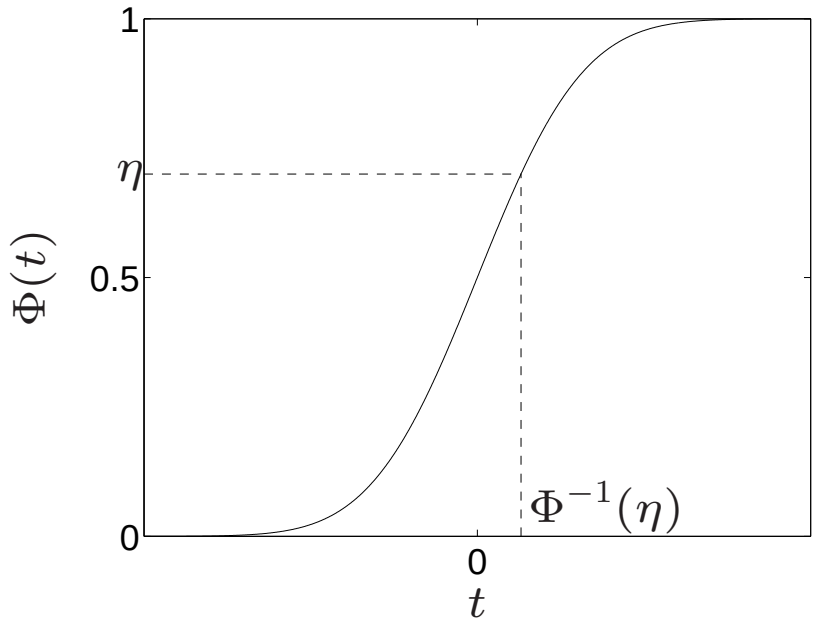
$\eta = 90\%$

# SOCP formulation

the 'chance constraint' $\mathbf{prob}(a_i^T x \leq b_i) \geq \eta$ is equivalent to the constraint

$$\bar{a}_i^T x + \Phi^{-1}(\eta)\|\Sigma_i^{1/2} x\|_2 \leq b_i$$

$\Phi$ is the (unit) normal cumulative density function




robust LP is a second-order cone program for $\eta \geq 0.5$

# Robust linear program (deterministic)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i \text{ for all } a_i \in \mathcal{E}_i, \quad i = 1, \ldots, m \end{array}$$

- $a_i$ uncertain but bounded by ellipsoid $\mathcal{E}_i = \{\bar{a}_i + P_i u \mid \|u\|_2 \leq 1\}$
- we require that $x$ satisfies each constraint for all possible $a_i$

## SOCP formulation

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \|P_i^T x\|_2 \leq b_i, \quad i = 1, \ldots, m \end{array}$$

follows from

$$\sup_{\|u\|_2 \leq 1} (\bar{a}_i + P_i u)^T x = \bar{a}_i^T x + \|P_i^T x\|_2$$

# Examples of second-order cone constraints

**convex quadratic constraint** ($A = LL^T$ positive definite)

$$x^T Ax + 2b^T x + c \leq 0$$

$$\Updownarrow$$

$$\left\| L^T x + L^{-1} b \right\|_2 \leq (b^T A^{-1} b - c)^{1/2}$$

extends to positive semidefinite singular $A$

**hyperbolic constraint**

$$x^T x \leq yz, \quad y, z \geq 0$$

$$\Updownarrow$$

$$\left\| \begin{bmatrix} 2x \\ y - z \end{bmatrix} \right\|_2 \leq y + z, \quad y, z \geq 0$$

# Examples of SOC-representable constraints

**positive powers**

$$x^{1.5} \leq t, \quad x \geq 0$$

$$\Updownarrow$$

$$\exists z: \quad x^2 \leq tz, \quad z^2 \leq x, \quad x, z \geq 0$$

- two hyperbolic constraints can be converted to SOC constraints
- extends to powers $x^p$ for rational $p \geq 1$

**negative powers**

$$x^{-3} \leq t, \quad x > 0$$

$$\Updownarrow$$

$$\exists z: \quad 1 \leq tz, \quad z^2 \leq tx, \quad x, z \geq 0$$

- two hyperbolic constraints on r.h.s. can be converted to SOC constraints
- extends to powers $x^p$ for rational $p < 0$

# Semidefinite program (SDP)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x_1 A_1 + x_2 A_2 + \cdots + x_n A_n \preceq B \end{array}$$

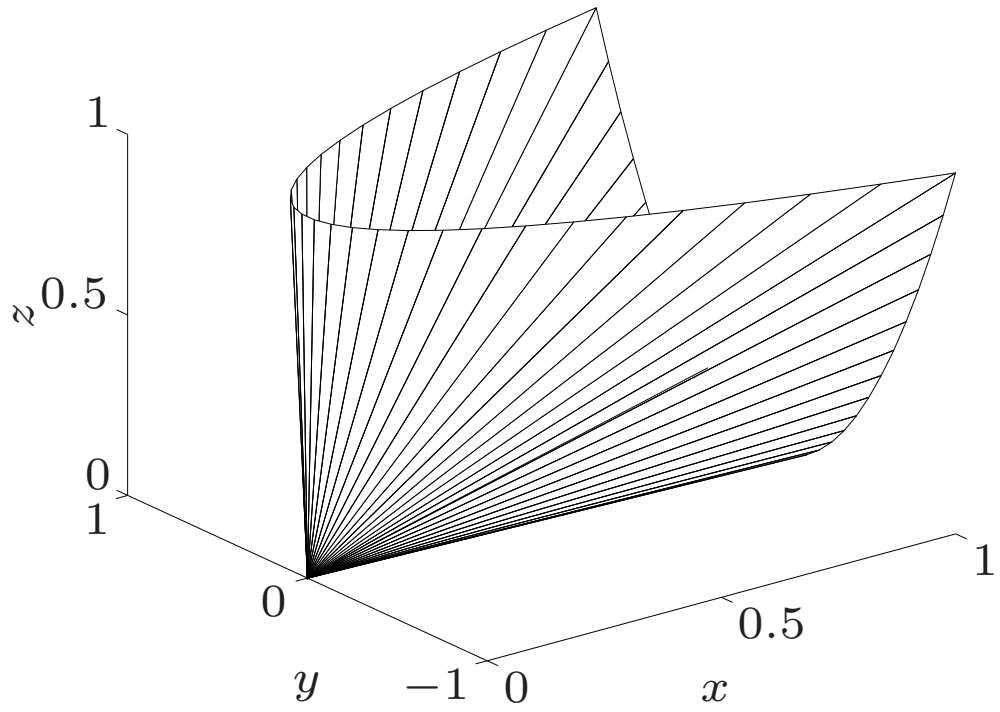
- $A_1$, $A_2$, . . . , $A_n$, $B$ are symmetric matrices
- inequality $X \preceq Y$ means $Y - X$ is *positive semidefinite*, $i.e.$,

$$z^T(Y - X)z = \sum_{i,j}(Y_{ij} - X_{ij})z_i z_j \geq 0 \text{ for all } z$$

- includes many nonlinear constraints as special cases

# Geometry

$$\begin{bmatrix} x & y \\ y & z \end{bmatrix} \succeq 0$$




- a nonpolyhedral convex cone
- feasible set of a semidefinite program is the intersection of the positive semidefinite cone in high dimension with planes

# Examples

$$A(x) = A_0 + x_1 A_1 + \cdots + x_m A_m \qquad (A_i \in \mathbf{S}^n)$$

**eigenvalue minimization** (and equivalent SDP)

$$\begin{array}{ll} \text{minimize} & \lambda_{\max}(A(x)) \end{array}$$

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & A(x) \preceq tI \end{array}$$

**matrix-fractional function**

$$\begin{array}{ll} \text{minimize} & b^T A(x)^{-1} b \\ \text{subject to} & A(x) \succeq 0 \end{array}$$

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \left[ \begin{array}{cc} A(x) & b \\ b^T & t \end{array} \right] \succeq 0 \end{array}$$

# Matrix norm minimization

$$A(x) = A_0 + x_1A_1 + x_2A_2 + \cdots + x_nA_n \qquad (A_i \in \mathbf{R}^{p\times q})$$

**matrix norm approximation** ($\|X\|_2 = \max_k \sigma_k(X)$)

minimize $\quad \|A(x)\|_2$

minimize $\quad t$

subject to $\quad \begin{bmatrix} tI & A(x)^T \\ A(x) & tI \end{bmatrix} \succeq 0$

**nuclear norm approximation** ($\|X\|_* = \sum_k \sigma_k(X)$)

minimize $\quad \|A(x)\|_*$

minimize $\quad (\mathbf{tr}\, U + \mathbf{tr}\, V)/2$

subject to $\quad \begin{bmatrix} U & A(x)^T \\ A(x) & V \end{bmatrix} \succeq 0$

# Semidefinite relaxations

semidefinite programming is often used

- to find good bounds for nonconvex polynomial problems, via **relaxation**
- as a heuristic for good suboptimal points

**example: Boolean least-squares**

$$\begin{array}{ll} \text{minimize} & \|Ax-b\|_2^2 \\ \text{subject to} & x_i^2 = 1, \quad i = 1, \ldots, n \end{array}$$

- basic problem in digital communications
- could check all $2^n$ possible values of $x \in \{-1, 1\}^n$ . . .
- an NP-hard problem, and very hard in general

# Semidefinite lifting

## Boolean least-squares problem

$$\begin{array}{ll} \text{minimize} & x^T A^T A x - 2b^T A x + b^T b \\ \text{subject to} & x_i^2 = 1, \quad i = 1, \ldots, n \end{array}$$

**reformulation:** introduce new variable $Y = xx^T$

$$\begin{array}{ll} \text{minimize} & \mathbf{tr}(A^T A Y) - 2b^T A x + b^T b \\ \text{subject to} & Y = xx^T \\ & \mathbf{diag}(Y) = \mathbf{1} \end{array}$$

- cost function and second constraint are linear (in the variables $Y$, $x$)
- first constraint is nonlinear and nonconvex

. . . still a very hard problem

# Semidefinite relaxation

replace $Y = xx^T$ with weaker constraint $Y \succeq xx^T$ to obtain relaxation

$$\begin{array}{ll} \text{minimize} & \mathbf{tr}(A^T AY) - 2b^T Ax + b^T b \\ \text{subject to} & Y \succeq xx^T \\ & \mathbf{diag}(Y) = \mathbf{1} \end{array}$$

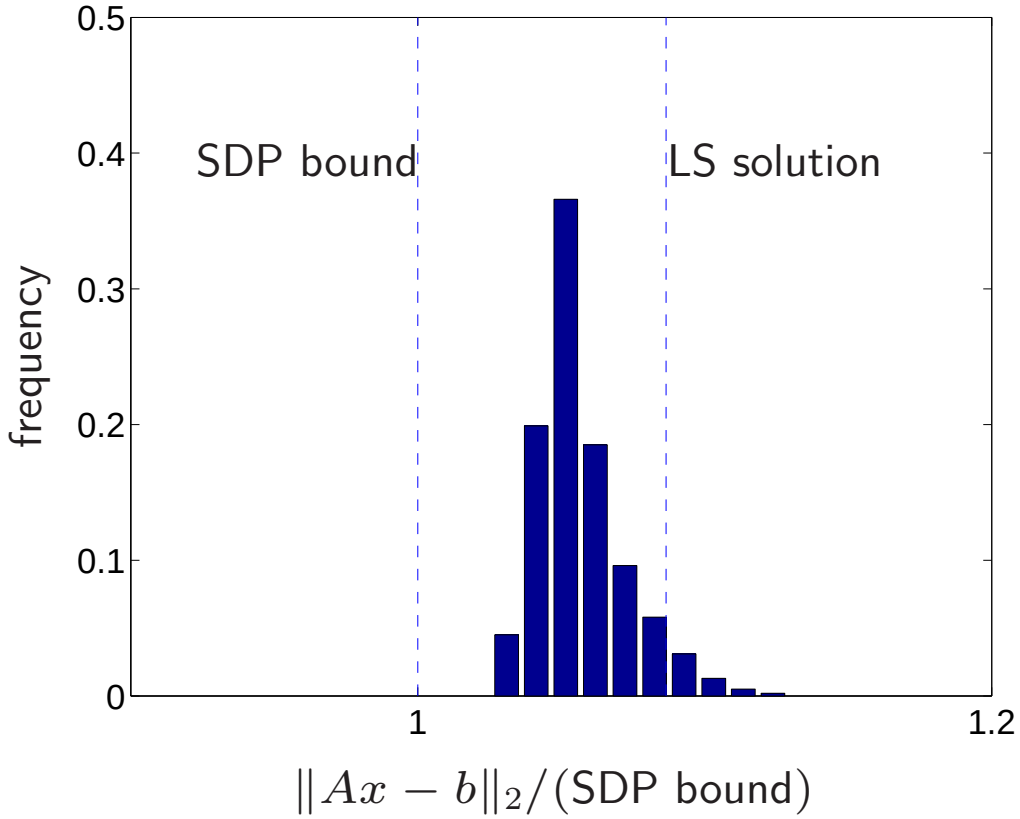
- convex; can be solved as a semidefinite program

$$Y \succeq xx^T \quad \Longleftrightarrow \quad \left[ \begin{array}{cc} Y & x \\ x^T & 1 \end{array} \right] \succeq 0$$

- optimal value gives lower bound for Boolean LS problem
- if $Y = xx^T$ at the optimum, we have solved the exact problem
- otherwise, can use *randomized rounding*

  generate $z$ from $\mathcal{N}(x, Y - xx^T)$ and take $x = \mathbf{sign}(z)$

# Example




- $n = 100$: feasible set has $2^{100} \approx 10^{30}$ points
- histogram of 1000 randomized solutions from SDP relaxation

# Nonnegative polynomial on R

$$f(t) = x_0 + x_1 t + \cdots + x_{2m} t^{2m} \geq 0 \quad \text{for all } t \in \mathbf{R}$$

- a convex constraint on $x$
- holds if and only if $f$ is a sum of squares of (two) polynomials:

$$\begin{aligned} f(t) &= \sum_k (y_{k0} + y_{k1} t + \cdots + y_{km} t^m)^2 \\ &= \begin{bmatrix} 1 \\ \vdots \\ t^m \end{bmatrix}^T \sum_k y_k y_k^T \begin{bmatrix} 1 \\ \vdots \\ t^m \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ \vdots \\ t^m \end{bmatrix}^T Y \begin{bmatrix} 1 \\ \vdots \\ t^m \end{bmatrix} \end{aligned}$$

where $Y = \sum_k y_k y_k^T \succeq 0$

# SDP formulation

$f(t) \geq 0$ if and only if for some $Y \succeq 0$,

$$f(t) = \begin{bmatrix} 1 \\ t \\ \vdots \\ t^{2m} \end{bmatrix}^T \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{2m} \end{bmatrix} = \begin{bmatrix} 1 \\ t \\ \vdots \\ t^{m} \end{bmatrix}^T Y \begin{bmatrix} 1 \\ t \\ \vdots \\ t^{m} \end{bmatrix}$$

this is an SDP constraint: there exists $Y \succeq 0$ such that

$$\begin{aligned} x_0 &= Y_{11} \\ x_1 &= Y_{12} + Y_{21} \\ x_2 &= Y_{13} + Y_{22} + Y_{32} \\ &\vdots \\ x_{2m} &= Y_{m+1,m+1} \end{aligned}$$

# General sum-of-squares constraints

$f(t) = x^T p(t)$ is a sum of squares if

$$x^T p(t) = \sum_{k=1}^{s} (y_k^T q(t))^2 = q(t)^T \left( \sum_{k=1}^{s} y_k y_k^T \right) q(t)$$

- $p$, $q$: basis functions (of polynomials, trigonometric polynomials, . . . )
- independent variable $t$ can be one- or multidimensional
- a *sufficient* condition for nonnegativity of $x^T p(t)$, useful in nonconvex polynomial optimization in several variables
- in some nontrivial cases ($e.g.$, polynomial on **R**), *necessary and sufficient*

**equivalent SDP constraint** (on the variables $x$, $X$)

$$x^T p(t) = q(t)^T X q(t), \qquad X \succeq 0$$

# Modeling software

**modeling packages for convex optimization**

- CVX, YALMIP (MATLAB)
- CVXMOD, CVXPY (Python)

assist in formulating convex problems by automating two tasks:

- verifying convexity from convex calculus rules
- transforming problem in input format required by standard solvers

**related packages**

general-purpose optimization modeling: AMPL, GAMS

# CVX example

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_1 \\ \text{subject to} & 0 \leq x_k \leq 1, \quad k = 1, \ldots, n \end{array}$$

## MATLAB code

```
cvx_begin
    variable x(3);
    minimize(norm(A*x - b, 1))
    subject to
        x >= 0;
        x <= 1;
cvx_end
```

- between `cvx_begin` and `cvx_end`, `x` is a CVX variable
- after execution, `x` is MATLAB variable with optimal solution

# Conic optimization

- definitions and examples
- modeling
- duality

# Generalized (conic) inequalities

**conic inequality:** a constraint $x \in K$ with $K$ a convex cone in $\mathbf{R}^m$

we require that $K$ is a **proper** cone:

- closed
- pointed: $K \cap (-K) = \{0\}$
- with nonempty interior: $\mathbf{int}\, K \neq \emptyset$; equivalently, $K + (-K) = \mathbf{R}^m$

**notation**

$$x \succeq_K y \iff x - y \in K, \qquad x \succ_K y \iff x - y \in \mathbf{int}\, K$$

with subscript in $\succeq_K$ omitted if $K$ is clear from the context

# Cone linear program

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq_K b \end{array}$$

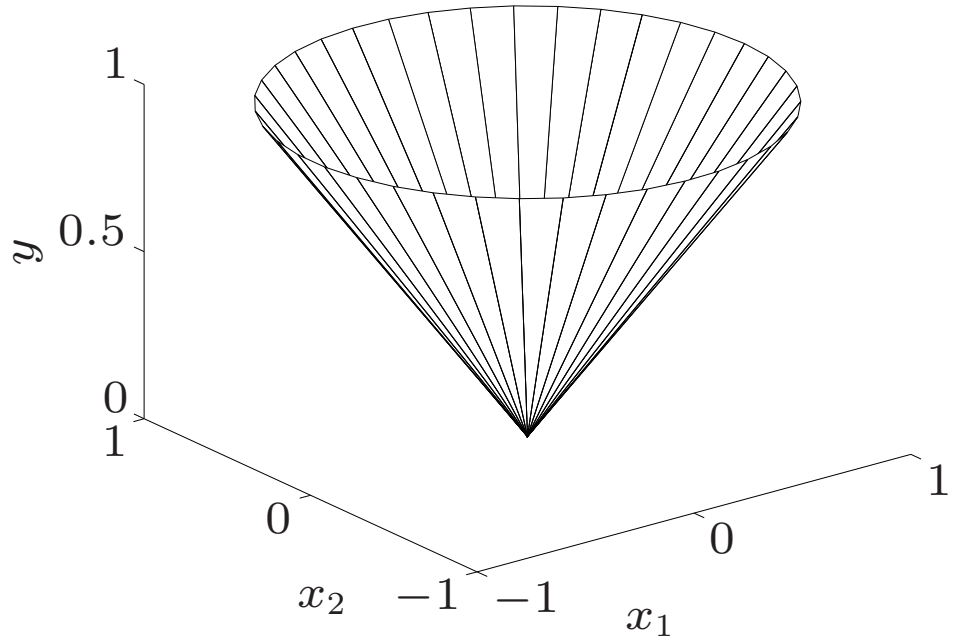
if $K$ is the nonnegative orthant, this reduces to regular linear program

widely used in recent literature on convex optimization

- **modeling:** a small number of 'primitive' cones is sufficient to express most convex constraints that arise in practice
- **algorithms**: a convenient problem format for extending interior-point algorithms for linear programming to convex optimization

# Norm cones

$$K = \left\{(x, y) \in \mathbf{R}^{m-1} \times \mathbf{R} \mid \|x\| \leq y\right\}$$




for the Euclidean norm this is the second-order cone (notation: $\mathcal{Q}^m$)

# Second-order cone program

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \|B_{k0}x + d_{k0}\|_2 \leq B_{k1}x + d_{k1}, \quad k = 1, \ldots, r \end{array}$$

**cone LP formulation:** express constraints as $Ax \preceq_K b$

$$K = \mathcal{Q}^{m_1} \times \cdots \times \mathcal{Q}^{m_r}, \qquad A = \begin{bmatrix} -B_{10} \\ -B_{11} \\ \vdots \\ -B_{r0} \\ -B_{r1} \end{bmatrix}, \qquad b = \begin{bmatrix} d_{10} \\ d_{11} \\ \vdots \\ d_{r0} \\ d_{r1} \end{bmatrix}$$

(assuming $B_{k0}$, $d_{k0}$ have $m_k - 1$ rows)

# Vector notation for symmetric matrices

- vectorized symmetric matrix: for $U \in \mathbf{S}^p$

$$\mathbf{vec}(U) = \sqrt{2}\left(\frac{U_{11}}{\sqrt{2}}, U_{21}, \ldots, U_{p1}, \frac{U_{22}}{\sqrt{2}}, U_{32}, \ldots, U_{p2}, \ldots, \frac{U_{pp}}{\sqrt{2}}\right)$$

- inverse operation: for $u = (u_1, u_2, \ldots, u_n) \in \mathbf{R}^n$ with $n = p(p+1)/2$

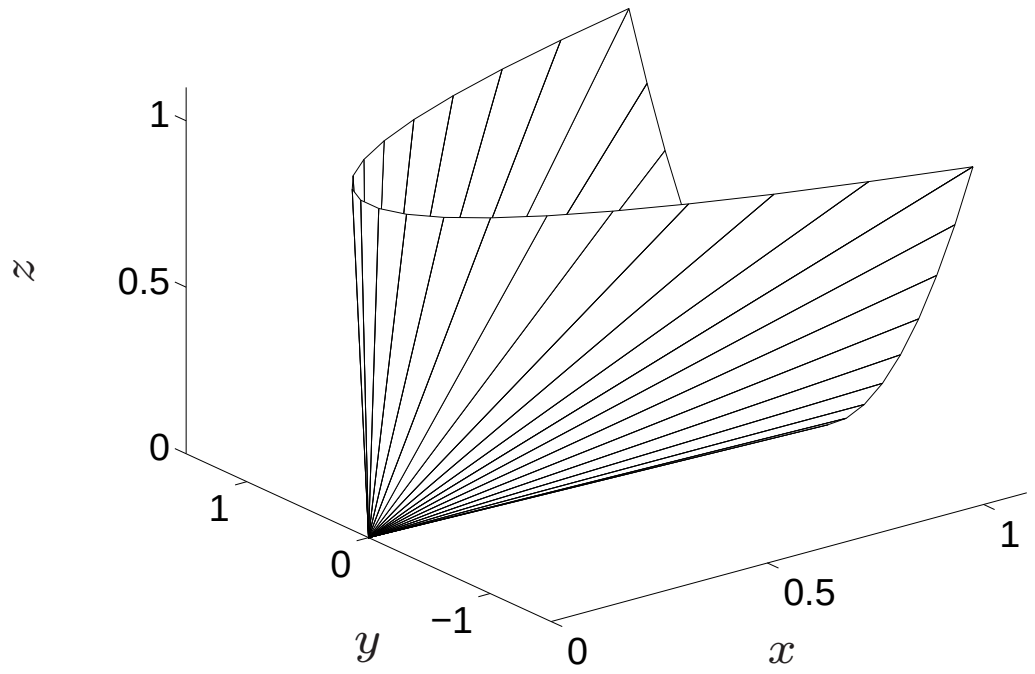
$$\mathbf{mat}(u) = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2}u_1 & u_2 & \cdots & u_p \\ u_2 & \sqrt{2}u_{p+1} & \cdots & u_{2p-1} \\ \vdots & \vdots & & \vdots \\ u_p & u_{2p-1} & \cdots & \sqrt{2}u_{p(p+1)/2} \end{bmatrix}$$

coefficients $\sqrt{2}$ are added so that standard inner products are preserved:

$$\mathbf{tr}(UV) = \mathbf{vec}(U)^T\,\mathbf{vec}(V), \qquad u^T v = \mathbf{tr}(\mathbf{mat}(u)\,\mathbf{mat}(v))$$

# Positive semidefinite cone

$$\mathcal{S}^p = \{\mathbf{vec}(X) \mid X \in \mathbf{S}_+^p\} = \{x \in \mathbf{R}^{p(p+1)/2} \mid \mathbf{mat}(x) \succeq 0\}$$




$$\mathcal{S}^2 = \left\{ (x, y, z) \;\middle|\; \begin{bmatrix} x & y/\sqrt{2} \\ y/\sqrt{2} & z \end{bmatrix} \succeq 0 \right\}$$

# Semidefinite program

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x_1 A_{11} + x_2 A_{12} + \cdots + x_n A_{1n} \preceq B_1 \\ & \cdots \\ & x_1 A_{r1} + x_2 A_{r2} + \cdots + x_n A_{rn} \preceq B_r \end{array}$$

$r$ linear matrix inequalities of order $p_1, \ldots, p_r$

**cone LP formulation:** express constraints as $Ax \preceq_K B$

$$K = \mathcal{S}^{p_1} \times \mathcal{S}^{p_2} \times \cdots \times \mathcal{S}^{p_r}$$

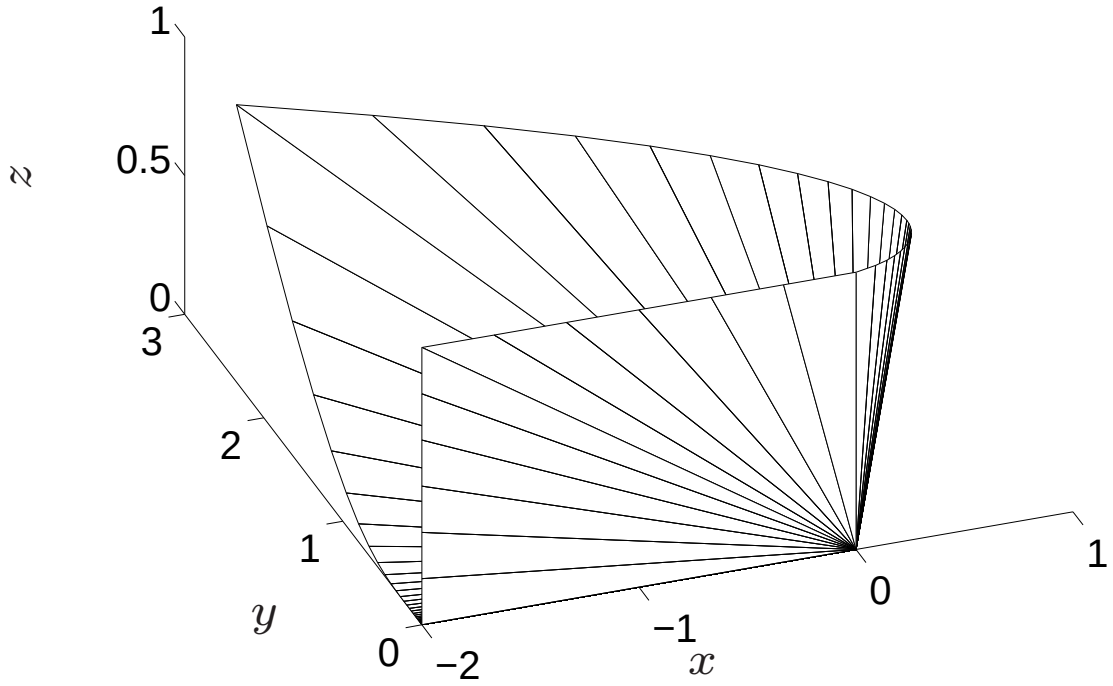
$$A = \begin{bmatrix} \mathbf{vec}(A_{11}) & \mathbf{vec}(A_{12}) & \cdots & \mathbf{vec}(A_{1n}) \\ \mathbf{vec}(A_{21}) & \mathbf{vec}(A_{22}) & \cdots & \mathbf{vec}(A_{2n}) \\ \vdots & \vdots & & \vdots \\ \mathbf{vec}(A_{r1}) & \mathbf{vec}(A_{r2}) & \cdots & \mathbf{vec}(A_{rn}) \end{bmatrix}, \qquad b = \begin{bmatrix} \mathbf{vec}(B_1) \\ \mathbf{vec}(B_2) \\ \vdots \\ \mathbf{vec}(B_r) \end{bmatrix}$$

# Exponential cone

the epigraph of the perspective of $\exp x$ is a non-proper cone

$$K = \left\{ (x, y, z) \in \mathbf{R}^3 \mid ye^{x/y} \leq z,\ y > 0 \right\}$$

the exponential cone is $K_{\exp} = \mathbf{cl}\, K = K \cup \{(x, 0, z) \mid x \leq 0, z \geq 0\}$



# Geometric program

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \log \sum_{k=1}^{n_i} \exp(a_{ik}^T x + b_{ik}) \leq 0, \quad i = 1, \ldots, r \end{array}$$

## cone LP formulation

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \begin{bmatrix} a_{ik}^T x + b_{ik} \\ 1 \\ z_{ik} \end{bmatrix} \in K_{\exp}, \quad k = 1, \ldots, n_i, \quad i = 1, \ldots, r \\ & \sum_{k=1}^{n_i} z_{ik} \leq 1, \quad i = 1, \ldots, m \end{array}$$

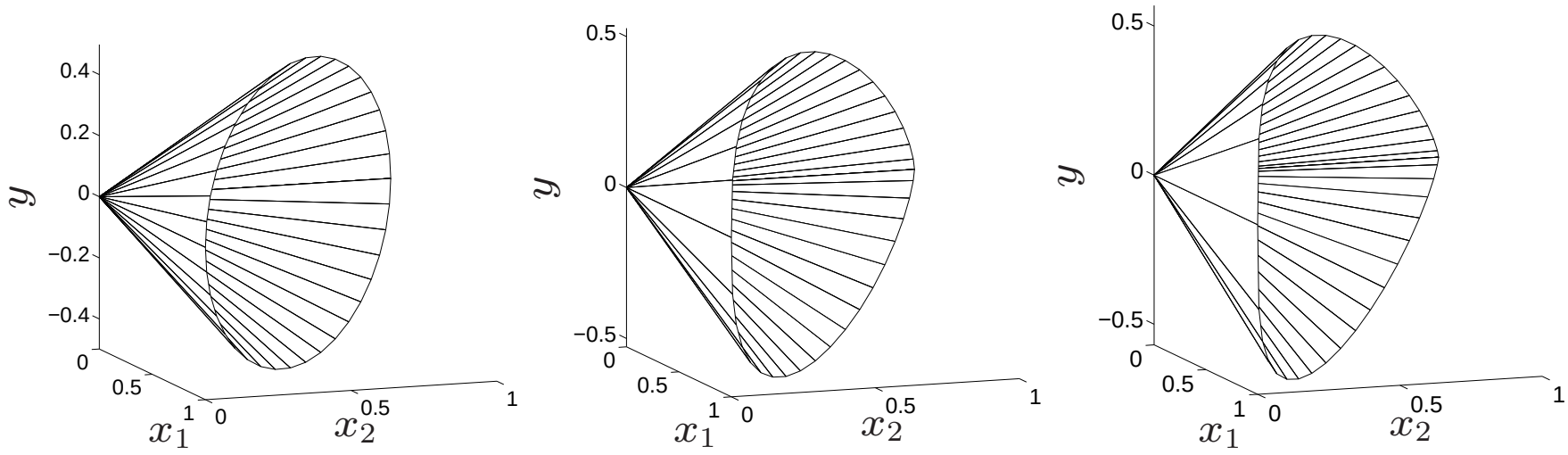
# Power cone

**definition:** for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) > 0$, $\sum_{i=1}^{m} \alpha_i = 1$

$$K_\alpha = \left\{ (x, y) \in \mathbf{R}_+^m \times \mathbf{R} \mid |y| \leq x_1^{\alpha_1} \cdots x_m^{\alpha_m} \right\}$$

**examples** for $m = 2$

$\alpha = (\frac{1}{2}, \frac{1}{2})$ $\qquad$ $\alpha = (\frac{2}{3}, \frac{1}{3})$ $\qquad$ $\alpha = (\frac{3}{4}, \frac{1}{4})$



# Outline

- definition and examples
- **modeling**
- duality

# Modeling tools

**convex modeling systems** (CVX, YALMIP, CVXMOD, CVXPY, . . . )

- convert problems stated in standard mathematical notation to cone LPs
- in principle, any convex problem can be represented as a cone LP
- in practice, a small set of primitive cones is used ($\mathbf{R}^n_+$, $\mathcal{Q}^p$, $\mathcal{S}^p$)
- choice of cones is limited by available algorithms and solvers (see later)

modeling systems implement set of rules for expressing constraints

$$f(x) \leq t$$

as conic inequalities for the implemented cones

# Examples of second-order cone representable functions

- convex quadratic

$$f(x) = x^T P x + q^T x + r \qquad (P \succeq 0)$$

- quadratic-over-linear function

$$f(x,y) = \frac{x^T x}{y} \quad \text{with } \mathbf{dom}\, f = \mathbf{R}^n \times \mathbf{R}_+ \qquad (\text{assume } 0/0 = 0)$$

- convex powers with rational exponent

$$f(x) = |x|^\alpha, \qquad f(x) = \begin{cases} x^\beta & x > 0 \\ +\infty & x \leq 0 \end{cases}$$

  for rational $\alpha \geq 1$ and $\beta \leq 0$

- $p$-norm $f(x) = \|x\|_p$ for rational $p \geq 1$

# Examples of SD cone representable functions

- matrix-fractional function

$$f(X,y) = y^T X^{-1} y \quad \text{with } \mathbf{dom}\, f = \{(X,y) \in \mathbf{S}_+^n \times \mathbf{R}^n \mid y \in \mathcal{R}(X)\}$$

- maximum eigenvalue of symmetric matrix

- maximum singular value $f(X) = \|X\|_2 = \sigma_1(X)$

$$\|X\|_2 \leq t \quad \Longleftrightarrow \quad \left[\begin{array}{cc} tI & X \\ X^T & tI \end{array}\right] \succeq 0$$

- nuclear norm $f(X) = \|X\|_* = \sum_i \sigma_i(X)$

$$\|X\|_* \leq t \quad \Longleftrightarrow \quad \exists U, V : \left[\begin{array}{cc} U & X \\ X^T & V \end{array}\right] \succeq 0, \quad \frac{1}{2}(\mathbf{tr}\, U + \mathbf{tr}\, V) \leq t$$

# Functions representable with exponential and power cone

## exponential cone

- exponential and logarithm
- entropy $f(x) = x \log x$

## power cone

- increasing power of absolute value: $f(x) = |x|^p$ with $p \geq 1$
- decreasing power: $f(x) = x^q$ with $q \leq 0$ and domain $\mathbf{R}_{++}$
- $p$-norm: $f(x) = \|x\|_p$ with $p \geq 1$

# Outline

- definition and examples
- modeling
- **duality**

# Linear programming duality

**primal and dual LP**

$$\begin{array}{lll} \text{(P)} & \text{minimize} & c^T x \\ & \text{subject to} & Ax \leq b \end{array} \qquad \begin{array}{lll} \text{(D)} & \text{maximize} & -b^T z \\ & \text{subject to} & A^T z + c = 0 \\ & & z \geq 0 \end{array}$$

- primal optimal value is $p^\star$ ($+\infty$ if infeasible, $-\infty$ if unbounded below)
- dual optimal value is $d^\star$ ($-\infty$ if infeasible, $+\infty$ if unbounded below)

**duality theorem**

- weak duality: $p^\star \geq d^\star$, with no exception
- strong duality: $p^\star = d^\star$ if primal or dual is feasible
- if $p^\star = d^\star$ is finite, then primal and dual optima are attained

# Dual cone

**definition**

$$K^* = \{y \mid x^T y \geq 0 \text{ for all } x \in K\}$$

a proper cone if $K$ is a proper cone

**dual inequality:** $x \succeq_* y$ means $x \succeq_{K^*} y$ for generic proper cone $K$

note: dual cone depends on choice of inner product:

$$H^{-1}K^*$$

is dual cone for inner product $\langle x, y\rangle = x^T H y$

# Examples

- $\mathbf{R}^p_+$, $\mathcal{Q}^p$, $\mathcal{S}^p$ are self-dual: $K = K^*$
- dual of norm cone is norm cone for dual norm
- dual of exponential cone

$$K^*_{\exp} = \left\{(u, v, w) \in \mathbf{R}_- \times \mathbf{R} \times \mathbf{R}^+ \mid -u \log(-u/w) + u - v \leq 0\right\}$$

(with $0 \log(0/w) = 0$ if $w \geq 0$)

- dual of power cone is

$$K^*_\alpha = \left\{(u, v) \in \mathbf{R}^m_+ \times \mathbf{R} \mid |v| \leq (u_1/\alpha_1)^{\alpha_1} \cdots (u_m/\alpha_m)^{\alpha_m}\right\}$$

# Primal and dual cone LP

**primal problem** (optimal value $p^\star$)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \end{array}$$

**dual problem** (optimal value $d^\star$)

$$\begin{array}{ll} \text{maximize} & -b^T z \\ \text{subject to} & A^T z + c = 0 \\ & z \succeq_* 0 \end{array}$$

**weak duality**: $p^\star \geq d^\star$ (without exception)

# Strong duality

$$p^\star = d^\star$$

if primal or dual is strictly feasible

- slightly weaker than LP duality (which only requires feasibility)
- can have $d^\star < p^\star$ with finite $p^\star$ and $d^\star$

**other implications of strict feasibility**

- if primal is strictly feasible, then dual optimum is attained (if $d^\star$ is finite)
- if dual is strictly feasible, then primal optimum is attained (if $p^\star$ is finite)

# Optimality conditions

| | | | |
|---|---|---|---|
| minimize | $c^T x$ | maximize | $-b^T z$ |
| subject to | $Ax + s = b$ | subject to | $A^T z + c = 0$ |
| | $s \succeq 0$ | | $z \succeq_* 0$ |

**optimality conditions**

$$\begin{bmatrix} 0 \\ s \end{bmatrix} = \begin{bmatrix} 0 & A^T \\ -A & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} c \\ b \end{bmatrix}$$

$$s \succeq 0, \qquad z \succeq_* 0, \qquad z^T s = 0$$

**duality gap:** inner product of $(x, z)$ and $(0, s)$ gives

$$z^T s = c^T x + b^T z$$

# Barrier methods

- barrier method for linear programming
- normal barriers
- barrier method for conic optimization

# History

- 1960s: Sequentially Unconstrained Minimization Technique (SUMT)

  solves nonlinear convex optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \end{array}$$

  via a sequence of unconstrained minimization problems

$$\text{minimize} \quad tf_0(x) - \sum_{i=1}^{m} \log(-f_i(x))$$

- 1980s: LP barrier methods with polynomial worst-case complexity
- 1990s: barrier methods for non-polyhedral cone LPs

# Logarithmic barrier function for linear inequalities

$$\psi(x) = \phi(b - Ax), \qquad \phi(s) = -\sum_{i=1}^{m} \log s_i$$

- a smooth convex function with $\mathbf{dom}\,\psi = \{x \mid Ax < b\}$
- $\psi(x) \to \infty$ at boundary of $\mathbf{dom}\,\psi$
- gradient and Hessian are

$$\nabla\psi(x) = -A^T\nabla\phi(s), \qquad \nabla^2\psi(x) = A^T\nabla\phi^2(s)A$$

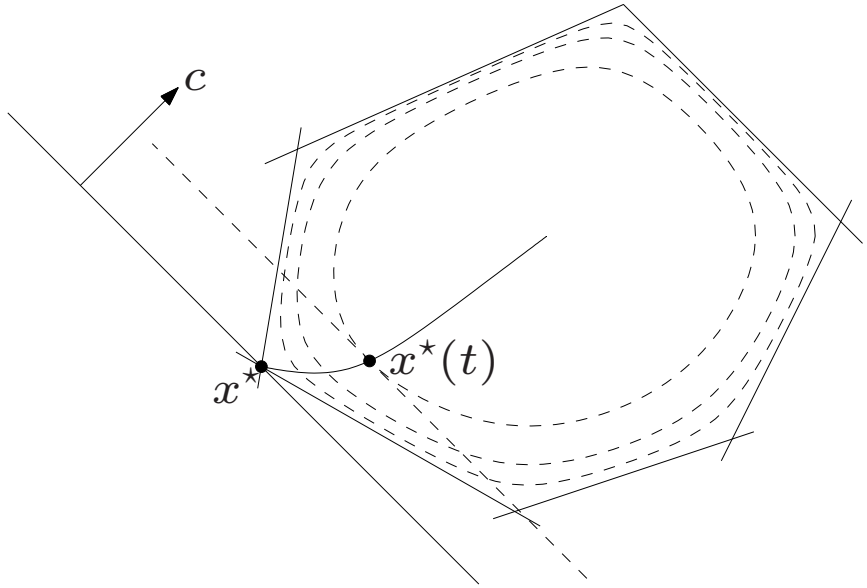
with $s = b - Ax$

$$\nabla\phi(s) = -\left(\frac{1}{s_1}, \ldots, \frac{1}{s_m}\right), \qquad \nabla\phi^2(s) = \mathbf{diag}\left(\frac{1}{s_1^2}, \ldots, \frac{1}{s_m^2}\right)$$

# Central path for linear program

**central path:** set of minimizers $x^\star(t)$ (with $t > 0$) of

$$f_t(x) = tc^T x + \phi(b - Ax)$$




**optimality conditions:** $x = x^\star(t)$ satisfies

$$\nabla f_t(x) = tc - A^T \nabla \phi(s) = 0, \qquad s = b - Ax$$

# Central path and duality

**dual feasible point on central path**

- for $x = x^\star(t)$ and $s = b - Ax$,

$$z^*(t) = -\frac{1}{t}\nabla\phi(s) = \left(\frac{1}{ts_1}, \frac{1}{ts_2}, \ldots, \frac{1}{ts_m}\right)$$

  is strictly dual feasible: $c + A^T z = 0$ and $z > 0$

- can be modified to correct for inexact centering of $x$

**duality gap** between $x = x^\star(t)$ and $z = z^\star(t)$ is

$$c^T x + b^T z = s^T z = \frac{m}{t}$$

gives bound on suboptimality: $c^T x^\star(t) - p^\star \leq m/t$

# Barrier method

starting with $t > 0$, strictly feasible $x$, repeat until $c^T x - p^\star \leq \epsilon$

- make one or more Newton steps to (approximately) minimize $f_t$:

$$x^+ = x - \alpha \nabla^2 f_t(x)^{-1} \nabla f_t(x)$$

  step size $\alpha$ is fixed or from line search

- increase $t$

**complexity:** with proper initialization, step size, update scheme for $t$,

$$\#\text{Newton steps} = O\left(\sqrt{m}\log(1/\epsilon)\right)$$

result follows from convergence analysis of Newton's method for $f_t$

# Outline

- barrier method for linear programming
- **normal barriers**
- barrier method for conic optimization

# Normal barrier for proper cone

$\phi$ is a $\theta$-normal barrier for the proper cone $K$ if it is

- a **barrier**: smooth, convex, domain $\mathbf{int}\, K$, blows up at boundary of $K$
- **logarithmically homogeneous** with parameter $\theta$:

$$\phi(tx) = \phi(x) - \theta \log t, \qquad \forall x \in \mathbf{int}\, K,\ t > 0$$

- **self-concordant**: restriction $g(\alpha) = \phi(x + \alpha v)$ to any line satisfies

$$g'''(\alpha) \leq 2g''(\alpha)^{3/2}$$

introduced by Nesterov and Nemirovski (1994)

# Examples

**nonnegative orthant:** $K = \mathbf{R}^m_+$

$$\phi(x) = -\sum_{i=1}^{m} \log x_i \qquad (\theta = m)$$

**second-order cone:** $K = \mathcal{Q}^p = \{(x, y) \in \mathbf{R}^{p-1} \times \mathbf{R} \mid \|x\|_2 \leq y\}$

$$\phi(x, y) = -\log(y^2 - x^T x) \qquad (\theta = 2)$$

**semidefinite cone**: $K = \mathcal{S}^m = \{x \in \mathbf{R}^{m(m+1)/2} \mid \mathbf{mat}(x) \succeq 0\}$

$$\phi(x) = -\log\det \mathbf{mat}(x) \qquad (\theta = m)$$

**exponential cone:** $K_{\exp} = \mathbf{cl}\{(x, y, z) \in \mathbf{R}^3 \mid ye^{x/y} \leq z,\ y > 0\}$

$$\phi(x, y, z) = -\log\left(y \log(z/y) - x\right) - \log z - \log y \qquad (\theta = 3)$$

**power cone:** $K = \{(x_1, x_2, y) \in \mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R} \mid |y| \leq x_1^{\alpha_1} x_2^{\alpha_2}\}$

$$\phi(x, y) = -\log\left(x_1^{2\alpha_1} x_2^{2\alpha_2} - y^2\right) - \log x_1 - \log x_2 \qquad (\theta = 4)$$

# Central path

**linear cone LP** (with inequality with respect to proper cone $K$)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \end{array}$$

**barrier for the feasible set**

$$\phi(b - Ax)$$

where $\phi$ is a $\theta$-normal barrier for $K$

**central path:** set of minimizers $x^\star(t)$ (with $t > 0$) of

$$f_t(x) = tc^T x + \phi(b - Ax)$$

# Newton step

**centering problem**

$$\text{minimize} \quad f_t(x) = tc^T x + \phi(b - Ax)$$

**Newton step** at $x$

$$\Delta x = -\nabla^2 f_t(x)^{-1} \nabla f_t(x)$$

**Newton decrement**

$$\begin{aligned} \lambda_t(x) &= \left(\Delta x^T \nabla^2 f_t(x) \Delta x\right)^{1/2} \\ &= \left(-\nabla f_t(x) \Delta x\right)^{1/2} \end{aligned}$$

used to measure proximity of $x$ to $x^\star(t)$

# Damped Newton method

$$\text{minimize} \quad f_t(x) = tc^T x + \phi(b - Ax)$$

**algorithm**

select $\epsilon \in (0, 1/2)$, $\eta \in (0, 1/4]$, and a starting point $x \in \mathbf{dom}\, f_t$

repeat:

1. compute Newton step $\Delta x$ and Newton decrement $\lambda_t(x)$
2. if $\lambda_t(x)^2 \leq \epsilon$, return $x$
3. otherwise, set $x := x + \alpha \Delta x$ with

$$\alpha = \frac{1}{1 + \lambda_t(x)} \quad \text{if } \lambda_t(x) \geq \eta, \qquad \alpha = 1 \quad \text{if } \lambda_t(x) < \eta$$

alternatively, can use backtracking line search

# Convergence results for damped Newton method

- **damped Newton phase**

$$f_t(x^+) - f_t(x) \leq -\gamma \quad \text{if } \lambda_t(x) \geq \eta$$

  value decreases by at least a positive constant $\gamma = \eta - \log(1+\eta)$

- **quadratically convergent phase**

$$2\lambda_t(x^+) \leq (2\lambda_t(x))^2 \quad \text{if } \lambda_t(x) < \eta$$

  implies $\lambda_t(x^+) \leq 2\eta^2 < \eta$, and Newton decrement decreases to zero

- **stopping criterion** $\lambda_t(x)^2 \leq \epsilon$ implies

$$f_t(x) - \inf f_t(x) \leq \epsilon$$

# Outline

- barrier method for linear programming
- normal barriers
- **barrier method for conic optimization**

# Central path and duality

**duality point on central path:** $x^\star(t)$ defines a strictly dual feasible $z^\star(t)$

$$z^\star(t) = -\frac{1}{t}\nabla\phi(s), \qquad s = b - Ax^\star(t)$$

**duality gap:** gap between $x = x^\star(t)$ and $z = z^\star(t)$ is

$$c^T x + b^T z = s^T z = \frac{\theta}{t}, \qquad c^T x - p^\star \leq \frac{\theta}{t}$$

**near central path:** for inexactly centered $x$

$$c^T x - p^\star \leq \left(1 + \frac{\lambda_t(x)}{\sqrt{\theta}}\right)\frac{\theta}{t} \qquad \text{if } \lambda_t(x) < 1$$

(results follow from properties of normal barriers)

# Short-step barrier method

**algorithm:** parameters $\epsilon \in (0,1)$, $\beta = 1/8$

- select initial $x$ and $t$ with $\lambda_t(x) \leq \beta$
- repeat until $2\theta/t \leq \epsilon$:

$$t := \left(1 + \frac{1}{1 + 8\sqrt{\theta}}\right) t, \qquad x := x - \nabla f_t(x)^{-1} \nabla f_t(x)$$

**properties**

- increase $t$ slowly so $x$ stays in region of quadratic region ($\lambda_t(x) \leq \beta$)
- iteration complexity

$$\#\text{iterations} = O\left(\sqrt{\theta} \log\left(\frac{\theta}{\epsilon t_0}\right)\right)$$

- best known worst-case complexity; same as for linear programming

# Predictor-corrector methods

**short-step barrier methods**

- stay in narrow neighborhood of central path (defined by limit on $\lambda_t$)
- make small, fixed increases $t^+ = \mu t$

as a result, quite slow in practice

**predictor-corrector method**

- select new $t$ using a linear approximation to central path ('predictor')
- re-center with new $t$ ('corrector')

allows faster and 'adaptive' increases in $t$; similar worst-case complexity

# Primal-dual methods

- primal-dual algorithms for linear programming
- symmetric cones
- primal-dual algorithms for conic optimization
- implementation

# Primal-dual interior-point methods

**similarities with barrier method**

- follow the same central path
- same linear algebra cost per iteration

**differences**

- more robust and faster (typically less than 50 iterations)
- primal and dual iterates updated at each iteration
- symmetric treatment of primal and dual iterates
- can start at infeasible points
- include heuristics for adaptive choice of central path parameter $t$
- often have superlinear asymptotic convergence

# Primal-dual central path for linear programming

| | | | |
|---|---|---|---|
| minimize | $c^T x$ | maximize | $-b^T z$ |
| subject to | $Ax + s = b$ | subject to | $A^T z + c = 0$ |
| | $s \geq 0$ | | $z \geq 0$ |

**optimality conditions**

$$Ax + s = b, \qquad A^T z + c = 0, \qquad (s, z) \geq 0, \qquad s \circ z = 0$$

$s \circ z$ is component-wise vector product

**primal-dual parametrization of central pah**

$$Ax + s = b, \qquad A^T z + c = 0, \qquad (s, z) \geq 0, \qquad s \circ z = \frac{1}{t}\mathbf{1}$$

solution is $x = x^*(t)$, $z = z^*(t)$

# Primal-dual search direction

steps solve central path equations linearized around current iterates $x$, $s$, $z$

$$A(x+\Delta x)+s+\Delta s=b \quad A^T(z+\Delta z)+c=0 \qquad (1)$$
$$(s+\Delta z)\circ(z+\Delta z)=\sigma\mu\mathbf{1}$$

where $\mu=(s^Tz)/m$ and $\sigma\in[0,1]$

- targets point on central path with $1/t=\sigma\mu$, *i.e.*, with gap $\sigma s^Tz$
- different methods use different strategies for selecting $\sigma$

**linearized equations:** the two linear equations in (1) and

$$z\circ\Delta s+s\circ\Delta z=\sigma\mu\mathbf{1}-s\circ z$$

after eliminating $\Delta s$, $\Delta z$ reduces to an equation

$$A^TDA\Delta x=r, \qquad D=\mathbf{diag}(z_1/s_1,\ldots,z_m/s_m)$$

# Outline

- primal-dual algorithms for linear programming
- **symmetric cones**
- primal-dual algorithms for conic optimization
- implementation

# Symmetric cones

symmetric primal-dual solvers for cone LPs are limited to **symmetric** cones

- second-order cone
- positive semidefinite cone
- direct products of these 'primitive' symmetric cones (such as $\mathbf{R}_+^p$)

**definition:** cone of squares $x = y^2 = y \circ y$ for a product $\circ$ that satisfies

1. bilinearity ($x \circ y$ is linear in $x$ for fixed $y$ and vice-versa)
2. $x \circ y = y \circ x$
3. $x^2 \circ (y \circ x) = (x^2 \circ y) \circ x$
4. $x^T(y \circ z) = (x \circ y)^T z$

not necessarily associative

# Vector product and identity element

**nonnegative orthant:** componentwise product

$$x \circ y = \mathbf{diag}(x)y$$

identity element is $\mathbf{e} = \mathbf{1} = (1, 1, \ldots, 1)$

**positive semidefinite cone**: symmetrized matrix product

$$x \circ y = \frac{1}{2}\,\mathbf{vec}(XY + YX) \quad \text{with } X = \mathbf{mat}(x),\ Y = \mathbf{mat}(Y)$$

identity element is $\mathbf{e} = \mathbf{vec}(I)$

**second-order cone**: the product of $x = (x_0, x_1)$ and $y = (y_0, y_1)$ is

$$x \circ y = \frac{1}{\sqrt{2}} \begin{bmatrix} x^T y \\ x_0 y_1 + y_0 x_1 \end{bmatrix}$$

identity element is $\mathbf{e} = (\sqrt{2}, 0, \ldots, 0)$

# Classification

- symmetric cones are studied in the theory of Euclidean Jordan algebras
- all possible symmetric cones have been characterized

**list of symmetric cones**

- the second-order cone
- the positive semidefinite cone of Hermitian matrices with real, complex, or quaternion entries
- $3 \times 3$ positive semidefinite matrices with octonion entries
- Cartesian products of these 'primitive' symmetric cones (such as $\mathbf{R}_+^p$)

**practical implication**

can focus on $\mathcal{Q}^p$, $\mathcal{S}^p$ and study these cones using elementary linear algebra

# Spectral decomposition

with each symmetric cone/product we associate a 'spectral' decomposition

$$x = \sum_{i=1}^{\theta} \lambda_i q_i, \qquad \text{with} \quad \sum_{i=1}^{\theta} q_i = \mathbf{e} \quad \text{and} \quad q_i \circ q_j = \begin{cases} q_i & i = j \\ 0 & i \neq j \end{cases}$$

**semidefinite cone** ($K = \mathcal{S}^p$): eigenvalue decomposition of $\mathbf{mat}(x)$

$$\theta = p, \qquad \mathbf{mat}(x) = \sum_{i=1}^{p} \lambda_i v_i v_i^T, \qquad q_i = \mathbf{vec}(v_i v_i^T)$$

**second-order cone** ($K = \mathcal{Q}^p$)

$$\theta = 2, \qquad \lambda_i = \frac{x_0 \pm \|x_1\|_2}{\sqrt{2}}, \qquad q_i = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \pm x_1/\|x_1\|_2 \end{bmatrix}, \qquad i = 1, 2$$

# Applications

**nonnegativity**

$$x \succeq 0 \quad \Longleftrightarrow \quad \lambda_1, \ldots, \lambda_\theta \geq 0, \qquad\qquad x \succ 0 \quad \Longleftrightarrow \quad \lambda_1, \ldots, \lambda_\theta > 0$$

**powers** (in particular, inverse and square root)

$$x^\alpha = \sum_i \lambda_i^\alpha q_i$$

**log-det barrier**

$$\phi(x) = -\log\det x = -\sum_{i=1}^{\theta} \log \lambda_i$$

- a $\theta$-normal barrier
- gradient is $\nabla\phi(x) = -x^{-1}$

# Outline

- primal-dual algorithms for linear programming
- symmetric cones
- **primal-dual algorithms for conic optimization**
- implementation

# Symmetric parametrization of central path

**centering problem**

$$\text{minimize} \quad tc^T x + \phi(b - Ax)$$

**optimality conditions** (using $\nabla\phi(s) = -s^{-1}$)

$$Ax + s = b, \qquad A^T z + c = 0, \qquad (s, z) \succ 0, \qquad z = \frac{1}{t} s^{-1}$$

**equivalent symmetric form**

$$Ax + b = s, \qquad A^T z + c = 0, \qquad (s, z) \succ 0, \qquad s \circ z = \frac{1}{t}\mathbf{e}$$

# Scaling with Hessian

linear transformation with $H = \nabla^2\phi(u)$ have several important properties

- preserves conic inequalities: $s \succ 0 \iff Hs \succ 0$
- if $s$ is invertible, then $Hs$ is invertible and $(Hs)^{-1} = H^{-1}s^{-1}$
- preserves central path:

$$s \circ z = \mu\,\mathbf{e} \quad \iff \quad (Hs) \circ (H^{-1}z) = \mu\,\mathbf{e}$$

- symmetric square root of $H$ is $H^{1/2} = \nabla^2\phi(u^{1/2})$

**example** ($K = \mathcal{S}^p$):

$$\tilde{S} = U^{-1}SU^{-1} \qquad S = \mathbf{mat}(s), \quad U = \mathbf{mat}(u)$$

# Primal-dual search direction

steps solve central path equations linearized around current iterates $x$, $s$, $z$

$$A(x+\Delta x)+s+\Delta s=b, \qquad A^T(z+\Delta z)+c=0 \tag{2}$$
$$(H(s+\Delta s))\circ\left(H^{-1}(z+\Delta z)\right)=\sigma\mu\mathbf{e}$$

where $\mu=(s^Tz)/m$, $\sigma\in[0,1]$, and $H=\nabla^2\phi(u)$

- different algorithms use different choices of $\sigma$, $u$
- Nesterov-Todd scaling: $H=\nabla^2\phi(u)$ defined by $Hs=H^{-1}z$

**linearized equations:** the two linear equations (2) and

$$(Hs)\circ(H^{-1}\Delta z)+(H^{-1}z)\circ(H\Delta s)=\sigma\mu\mathbf{e}-(Hs)\circ(H^{-1}z)$$

after eliminating $\Delta s$, $\Delta z$, reduces to an equation

$$A^T\nabla^2\phi(w)A\Delta x=r, \qquad w=u^2$$

# Outline

- primal-dual algorithms for linear programming
- symmetric cones
- primal-dual algorithms for conic optimization
- **implementation**

# Software implementations

## General-purpose software for nonlinear convex optimization

- several high-quality packages (MOSEK, Sedumi, SDPT3, . . . )
- exploit sparsity to achieve scalability

## Customized implementations

- can exploit non-sparse types of problem structure
- often orders of magnitude faster than general-purpose solvers

# Example: $\ell_1$-regularized least-squares

$$\text{minimize} \quad \|Ax - b\|_2^2 + \|x\|_1$$

$A$ is $m \times n$ (with $m \leq n$) and dense

**quadratic program formulations**

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_2^2 + \mathbf{1}^T u \\ \text{subject to} & -u \leq x \leq u \end{array}$$

- coefficient of Newton system in interior-point method is

$$\left[ \begin{array}{cc} A^T A & 0 \\ 0 & 0 \end{array} \right] + \left[ \begin{array}{cc} D_1 + D_2 & D_2 - D_1 \\ D_2 - D_1 & D_1 + D_2 \end{array} \right] \qquad (D_1,\ D_2 \text{ positive diagonal})$$

- expensive ($O(n^3)$) for large $n$

**customized implementation**

- can reduce Newton equation to solution of a system

$$(AD^{-1}A^T + I)\Delta u = r$$

- cost per iteration is $O(m^2n)$

**comparison** (seconds on 2.83 Ghz Core 2 Quad machine)

| $m$ | $n$ | custom | general-purpose |
|---|---|---|---|
| 50 | 200 | 0.02 | 0.32 |
| 50 | 400 | 0.03 | 0.59 |
| 100 | 1000 | 0.12 | 1.69 |
| 100 | 2000 | 0.24 | 3.43 |
| 500 | 1000 | 1.19 | 7.54 |
| 500 | 2000 | 2.38 | 17.6 |

custom solver is CVXOPT; general-purpose solver is MOSEK

# Gradient methods

- gradient and subgradient method
- proximal gradient method
- fast proximal gradient methods

# Classical gradient method

to minimize a convex differentiable function $f$: choose $x^{(0)}$ and repeat

$$x^{(k)} = x^{(k-1)} - t_k \nabla f(x^{(k-1)}), \qquad k = 1, 2, \ldots$$

step size $t_k$ is constant or from line search

**advantages**

- every iteration is inexpensive
- does not require second derivatives

**disadvantages**

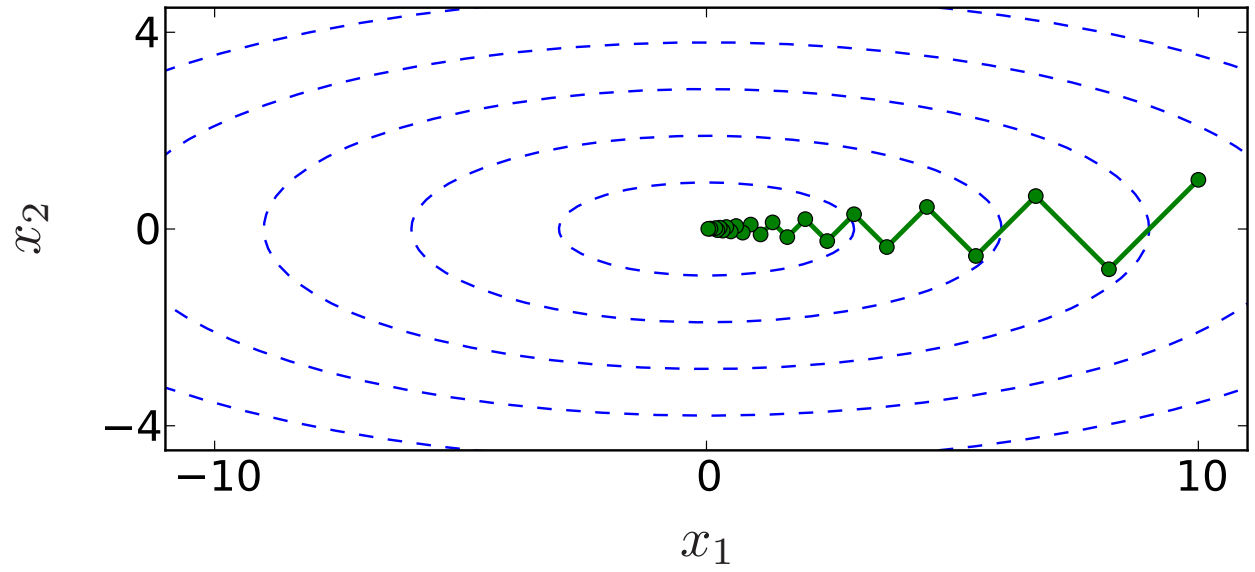
- often very slow; very sensitive to scaling
- does not handle nondifferentiable functions

# Quadratic example

$$f(x) = \frac{1}{2}(x_1^2 + \gamma x_2^2) \qquad (\gamma > 1)$$

with exact line search and starting point $x^{(0)} = (\gamma, 1)$

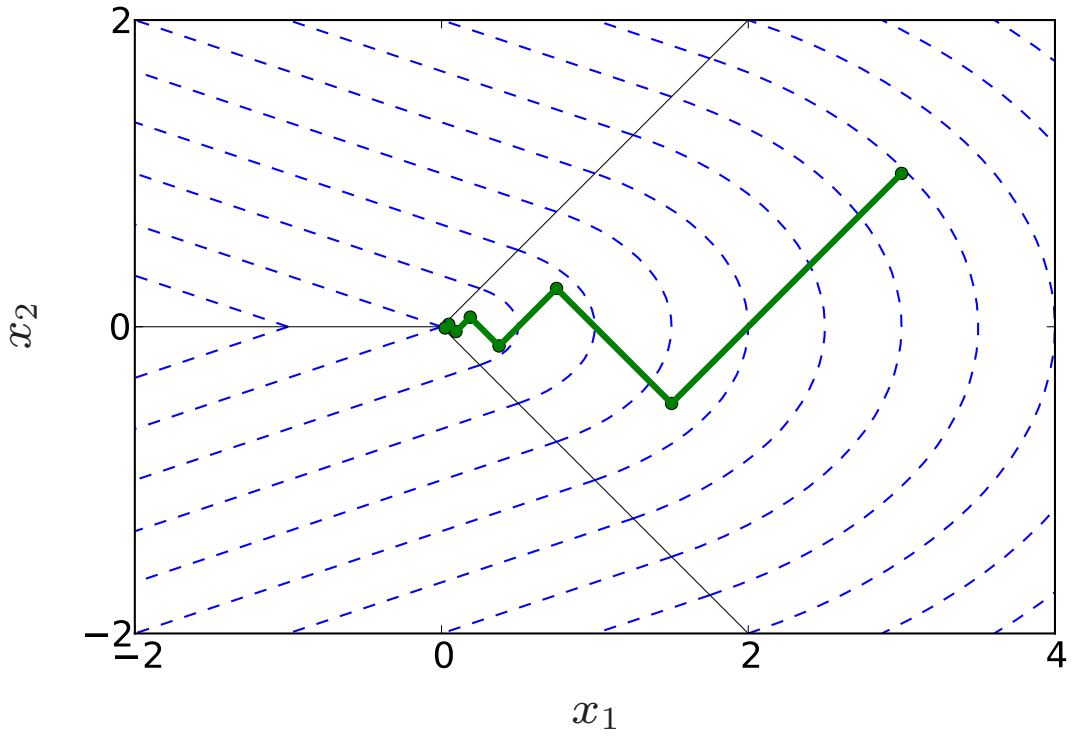
$$\frac{\|x^{(k)} - x^\star\|_2}{\|x^{(0)} - x^\star\|_2} = \left(\frac{\gamma - 1}{\gamma + 1}\right)^k$$



# Nondifferentiable example

$$f(x) = \sqrt{x_1^2 + \gamma x_2^2} \quad (|x_2| \leq x_1), \qquad f(x) = \frac{x_1 + \gamma |x_2|}{\sqrt{1+\gamma}} \quad (|x_2| > x_1)$$

with exact line search, $x^{(0)} = (\gamma, 1)$, converges to non-optimal point



# First-order methods

address one or both disadvantages of the gradient method

**methods for nondifferentiable or constrained problems**

- smoothing methods
- subgradient method
- proximal gradient method

**methods with improved convergence**

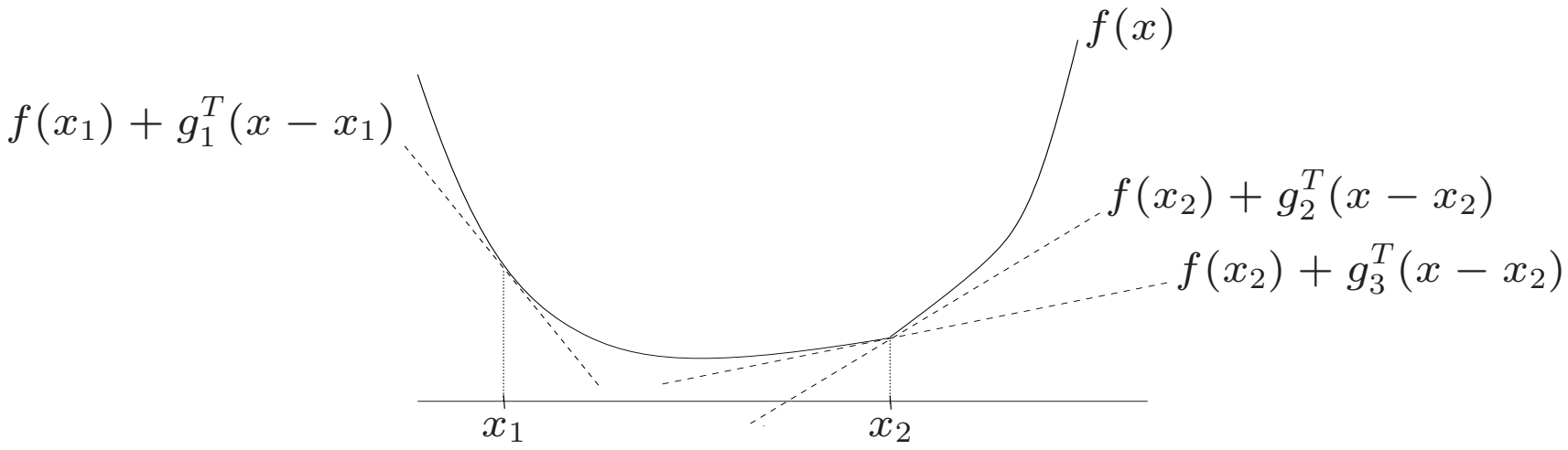
- variable metric methods
- conjugate gradient method
- accelerated proximal gradient method

we will discuss subgradient and proximal gradient methods

# Subgradient

$g$ is a subgradient of a convex function $f$ at $x$ if

$$f(y) \geq f(x) + g^T(y - x) \quad \forall y \in \mathbf{dom}\, f$$



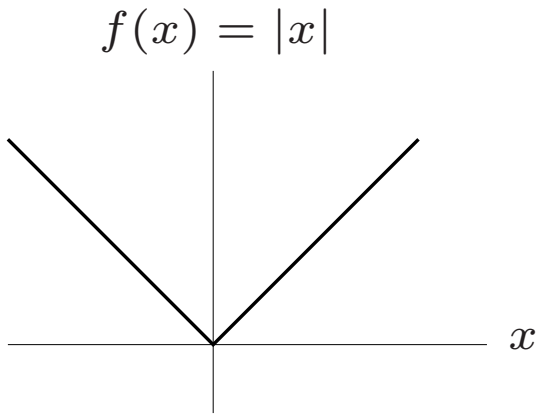

generalizes basic inequality for convex differentiable $f$

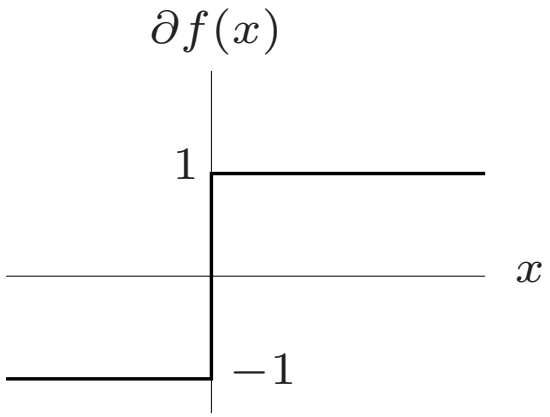
$$f(y) \geq f(x) + \nabla f(x)^T(y - x) \quad \forall y \in \mathbf{dom}\, f$$

# Subdifferential

the set of all subgradients of $f$ at $x$ is called the **subdifferential** $\partial f(x)$

**absolute value** $f(x) = |x|$




**Euclidean norm** $f(x) = \|x\|_2$

$$\partial f(x) = \frac{1}{\|x\|_2} x \quad \text{if } x \neq 0, \qquad \partial f(x) = \{g \mid \|g\|_2 \leq 1\} \quad \text{if } x = 0$$

# Subgradient calculus

**weak calculus**

rules for finding *one* subgradient

- sufficient for most algorithms for nondifferentiable convex optimization
- if one can evaluate $f(x)$, one can usually compute a subgradient
- much easier than finding the entire subdifferential

**subdifferentiability**

- convex $f$ is subdifferentiable on $\mathbf{dom}\, f$ except possibly at the boundary
- example of a non-subdifferentiable function: $f(x) = -\sqrt{x}$ at $x = 0$

# Examples of calculus rules

**nonnegative combination:** $f = \alpha_1 f_1 + \alpha_2 f_2$ with $\alpha_1, \alpha_2 \geq 0$

$$g = \alpha_1 g_1 + \alpha_2 g_2, \qquad g_1 \in \partial f_1(x), \quad g_2 \in \partial f_2(x)$$

**composition with affine transformation:** $f(x) = h(Ax + b)$

$$g = A^T \tilde{g}, \qquad \tilde{g} \in \partial h(Ax + b)$$

**pointwise maximum** $f(x) = \max\{f_1(x), \ldots, f_m(x)\}$

$$g \in \partial f_i(x) \quad \text{where } f_i(x) = \max_k f_k(x)$$

**conjugate** $f(x) = \sup_y (x^T y - f(y))$; take any maximizing $y$

# Subgradient method

to minimize a nondifferentiable convex function $f$: choose $x^{(0)}$ and repeat

$$x^{(k)} = x^{(k-1)} - t_k g^{(k-1)}, \quad k = 1, 2, \ldots$$

$g^{(k-1)}$ is **any** subgradient of $f$ at $x^{(k-1)}$

**step size rules**

- fixed step size: $t_k$ constant
- fixed step length: $t_k \|g^{(k-1)}\|_2$ constant ($i.e.$, $\|x^{(k)} - x^{(k-1)}\|_2$ constant)
- diminishing: $t_k \to 0$, $\sum_{k=1}^{\infty} t_k = \infty$

# Some convergence results

**assumption:** $f$ is convex and Lipschitz continuous with constant $G > 0$:

$$|f(x) - f(y)| \leq G\|x - y\|_2 \qquad \forall x, y$$

**results**

- fixed step size $t_k = t$

  converges to approximately $G^2 t/2$-suboptimal

- fixed length $t_k \|g^{(k-1)}\|_2 = s$

  converges to approximately $Gs/2$-suboptimal

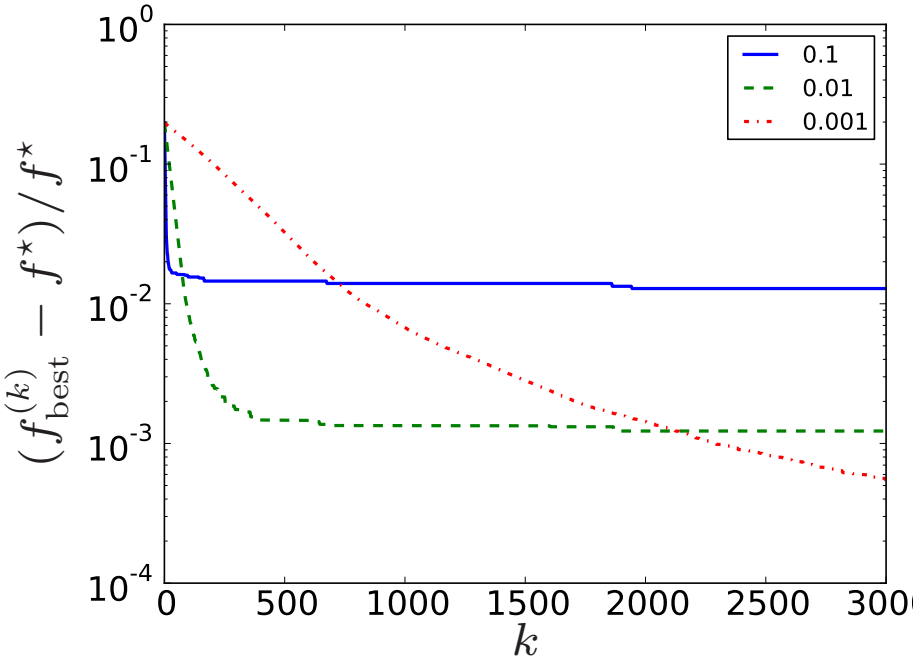
- decreasing $\sum_k t_k \to \infty$, $t_k \to 0$: convergence

  rate of convergence is $1/\sqrt{k}$ with proper choice of step size sequence

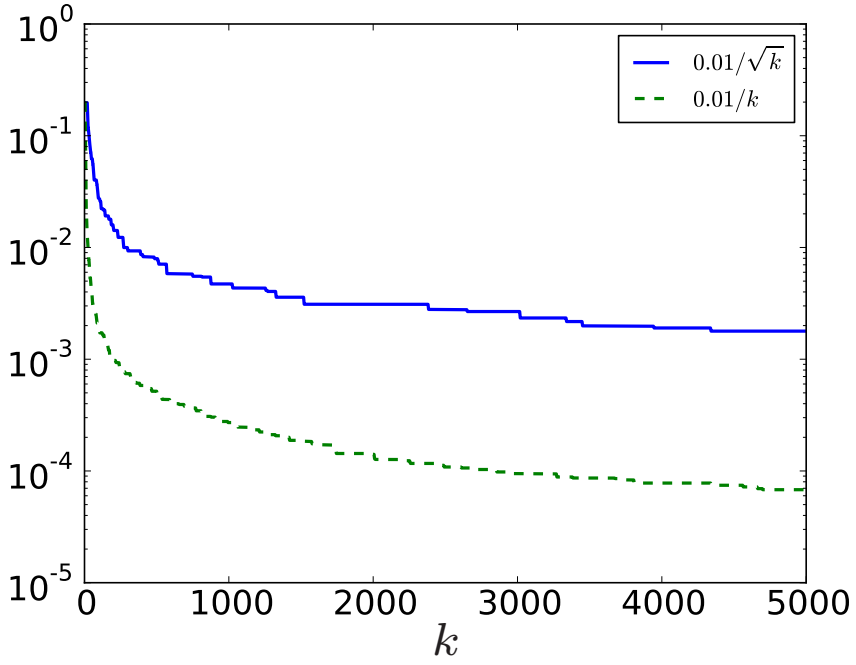
# Example: 1-norm minimization

minimize $\|Ax - b\|_1$ $\qquad (A \in \mathbf{R}^{500\times 100}, b \in \mathbf{R}^{500})$

subgradient is given by $A^T \mathbf{sign}(Ax - b)$




fixed steplength
$s = 0.1,\ 0.01,\ 0.001$




diminishing step size
$t_k = 0.01/\sqrt{k},\ t_k = 0.01/k$

# Outline

- gradient and subgradient method
- **proximal gradient method**
- fast proximal gradient methods

# Proximal mapping

the proximal mapping (prox-operator) of a convex function $h$ is

$$\mathrm{prox}_h(x) = \operatorname*{argmin}_u \left( h(u) + \frac{1}{2}\|u - x\|_2^2 \right)$$

- $h(x) = 0$: $\mathrm{prox}_h(x) = x$
- $h(x) = I_C(x)$ (indicator function of $C$): $\mathrm{prox}_h$ is projection on $C$

$$\mathrm{prox}_h(x) = \operatorname*{argmin}_{u \in C} \|u - x\|_2^2 = P_C(x)$$

- $h(x) = \|x\|_1$: $\mathrm{prox}_h$ is the 'soft-threshold' (shrinkage) operation

$$\mathrm{prox}_h(x)_i = \begin{cases} x_i - 1 & x_i \geq 1 \\ 0 & |x_i| \leq 1 \\ x_i + 1 & x_i \leq -1 \end{cases}$$

# Proximal gradient method

**unconstrained problem** with cost function split in two components

$$\text{minimize} \quad f(x) = g(x) + h(x)$$

- $g$ convex, differentiable, with $\mathbf{dom}\, g = \mathbf{R}^n$
- $h$ convex, possibly nondifferentiable, with inexpensive prox-operator

**proximal gradient algorithm**

$$x^{(k)} = \mathrm{prox}_{t_k h}\left(x^{(k-1)} - t_k \nabla g(x^{(k-1)})\right)$$

$t_k > 0$ is step size, constant or determined by line search

# Interpretation

$$x^+ = \mathrm{prox}_{th}\left(x - t\nabla g(x)\right)$$

from definition of proximal operator:

$$\begin{aligned} x^+ &= \underset{u}{\mathrm{argmin}} \left( h(u) + \frac{1}{2t}\|u - x + t\nabla g(x)\|_2^2 \right) \\ &= \underset{u}{\mathrm{argmin}} \left( h(u) + g(x) + \nabla g(x)^T(u-x) + \frac{1}{2t}\|u - x\|_2^2 \right) \end{aligned}$$

$x^+$ minimizes $h(u)$ plus a simple quadratic local model of $g(u)$ around $x$

# Examples

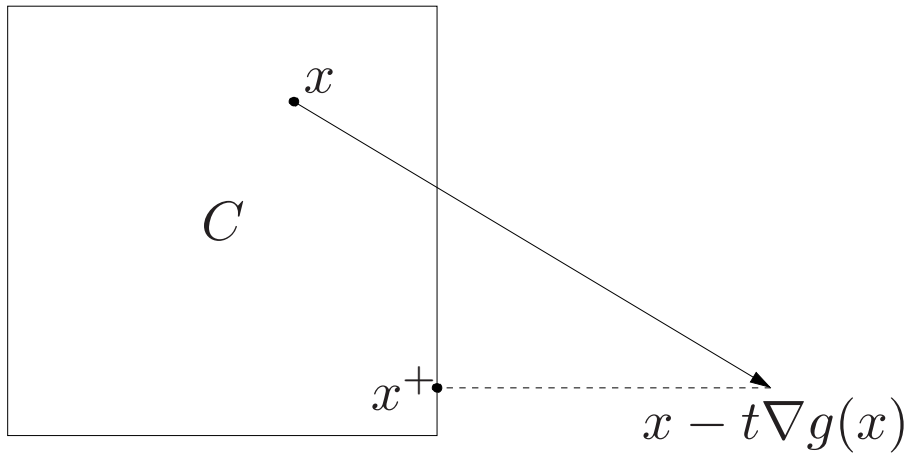
$$\text{minimize} \quad g(x) + h(x)$$

**gradient method:** $h(x) = 0$, *i.e.*, minimize $g(x)$

$$x^+ = x - t\nabla g(x)$$

**gradient projection method:** $h(x) = I_C(x)$, *i.e.*, minimize $g(x)$ over $C$

$$x^+ = P_C\left(x - t\nabla g(x)\right)$$

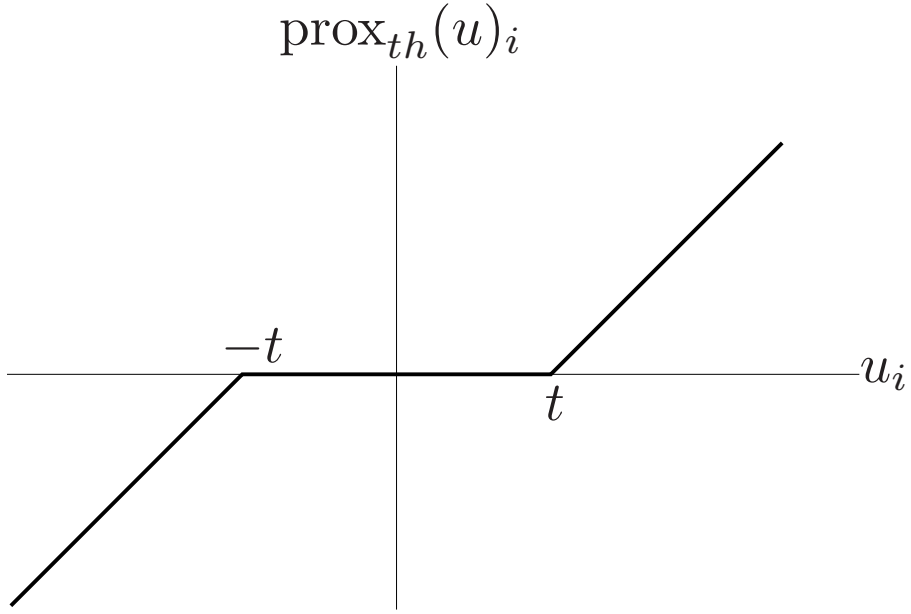


**iterative soft-thresholding:** $h(x) = \|x\|_1$, *i.e.*, minimize $g(x) + \|x\|_1$

$$x^+ = \mathrm{prox}_{th}\left(x - t\nabla g(x)\right)$$

and

$$\mathrm{prox}_{th}(u)_i = \begin{cases} u_i - t & u_i \geq t \\ 0 & -t \leq u_i \leq t \\ u_i + t & u_i \geq t \end{cases}$$



# Some properties of proximal mappings

$$\mathrm{prox}_h(x) = \operatorname*{argmin}_u \left( h(u) + \frac{1}{2}\|u - x\|_2^2 \right)$$

assume $h$ is closed and convex ($i.e.$, convex with closed epigraph)

- $\mathrm{prox}_h(x)$ is uniquely defined for all $x$
- $\mathrm{prox}_h$ is nonexpansive

$$\|\mathrm{prox}_h(x) - \mathrm{prox}_h(y)\|_2 \leq \|x - y\|_2$$

- Moreau decomposition

$$x = \mathrm{prox}_h(x) + \mathrm{prox}_{h^*}(x)$$

$cf.$, properties of Euclidean projection on convex sets

**example:** $h$ is indicator function of subspace $L$

$$h(u) = I_L(u) = \begin{cases} 0 & u \in L \\ +\infty & \text{otherwise} \end{cases}$$

- conjugate $h^*$ is indicator function of the orthogonal complement $L^\perp$

$$\begin{aligned} h^*(v) = \sup_{u \in L} v^T u &= \begin{cases} 0 & v \in L^\perp \\ +\infty & \text{otherwise} \end{cases} \\ &= I_{L^\perp}(v) \end{aligned}$$

- Moreau decomposition is orthogonal decomposition

$$x = P_L(x) + P_{L^\perp}(x)$$

# Examples of inexpensive prox-operators

**projection on simple sets**

- hyperplanes and halfspaces
- rectangles $\{x \mid l \leq x \leq u\}$
- probability simplex $\{x \mid \mathbf{1}^T x = 1, x \geq 0\}$
- norm ball for many norms (Euclidean, 1-norm, . . . )
- nonnegative orthant, second-order cone, positive semidefinite cone

**Euclidean norm:** $h(x) = \|x\|_2$

$$\mathrm{prox}_{th}(x) = \left(1 - \frac{t}{\|x\|_2}\right) x \quad \text{if } \|x\|_2 \geq t, \qquad \mathrm{prox}_{th}(x) = 0 \quad \text{otherwise}$$

**logarithmic barrier**

$$h(x) = -\sum_{i=1}^{n} \log x_i, \qquad \mathrm{prox}_{th}(x)_i = \frac{x_i + \sqrt{x_i^2 + 4t}}{2}, \quad i = 1, \ldots, n$$

**Euclidean distance:** $d(x) = \inf_{y \in C} \|x - y\|_2$ ($C$ closed convex)

$$\mathrm{prox}_{td}(x) = \theta P_C(x) + (1 - \theta)x, \qquad \theta = \frac{t}{\max\{d(x), t\}}$$

**squared Euclidean distance:** $h(x) = d(x)^2/2$

$$\mathrm{prox}_{th}(x) = \frac{1}{1+t}x + \frac{t}{1+t}P_C(x)$$

# Prox-operator of conjugate

$$\mathrm{prox}_{th^*}(x) = x - t\,\mathrm{prox}_{h/t}(x/t)$$

- follows from Moreau decomposition
- of interest when prox-operator of $h$ is inexpensive

**example: norms**

$$h(x) = I_C(x), \qquad h^*(y) = \|y\|_*$$

where $C$ is unit norm ball for $\|\cdot\|$ and $\|\cdot\|_*$ is dual norm of $\|\cdot\|$

- $\mathrm{prox}_h$ is projection on $C$
- formula useful for prox-operator of $\|\cdot\|_*$ if projection on $C$ is inexpensive

# Support function

many convex functions can be expressed as **support functions**

$$h(x) = S_C(x) = \sup_{y \in C} x^T y$$

with $C$ closed, convex

- conjugate is indicator function of $C$: $h^*(y) = I_C(y)$
- hence, can compute $\mathrm{prox}_{th}$ via projection on $C$

**example**: $h(x)$ is sum of largest $r$ components of $x$

$$h(x) = x_{[1]} + \cdots + x_{[r]} = S_C(x), \qquad C = \{y \mid 0 \leq y \leq \mathbf{1}, \mathbf{1}^T y = r\}$$

# Convergence of proximal gradient method

$$\text{minimize} \quad f(x) = g(x) + h(x)$$

**assumptions**

- $\nabla g$ is Lipschitz continuous with constant $L > 0$

$$\|\nabla g(x) - \nabla g(y)\|_2 \leq L\|x - y\|_2 \quad \forall x, y$$

- optimal value $f^\star$ is finite and attained at $x^\star$ (not necessarily unique)

**result:** with fixed step size $t_k = 1/L$

$$f(x^{(k)}) - f^\star \leq \frac{L}{2k}\|x^{(0)} - x^\star\|_2^2$$

- compare with $1/\sqrt{k}$ rate of subgradient method
- can be extended to account for line searches

# Outline

- gradient and subgradient method
- proximal gradient method
- **fast proximal gradient methods**

# Fast (proximal) gradient methods

- Nesterov (1983, 1988, 2005): three gradient projection methods with $1/k^2$ convergence rate
- Beck & Teboulle (2008): FISTA, a proximal gradient version of Nesterov's 1983 method
- Nesterov (2004 book), Tseng (2008): overview and unified analysis of fast gradient methods
- several recent variations and extensions

**this lecture:** FISTA (Fast Iterative Shrinkage-Thresholding Algorithm)

# FISTA

**unconstrained problem with composite objective**

$$\text{minimize} \quad f(x) = g(x) + h(x)$$

- $g$ convex differentiable with $\mathbf{dom}\, g = \mathbf{R}^n$
- $h$ convex with inexpensive prox-operator

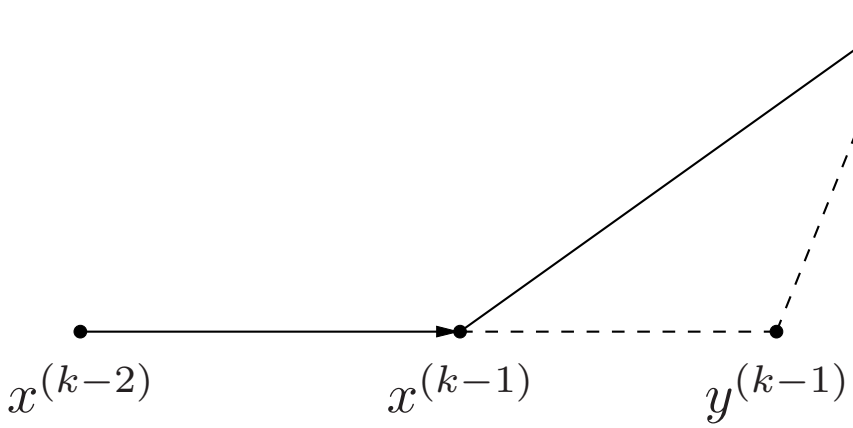
**algorithm:** choose $x^{(0)} = y^{(0)} \in \mathbf{dom}\, h$; for $k \geq 1$

$$\begin{aligned} x^{(k)} &= \mathrm{prox}_{t_k h}\left(y^{(k-1)} - t_k \nabla g(y^{(k-1)})\right) \\ y^{(k)} &= x^{(k)} + \frac{k-1}{k+2}(x^{(k)} - x^{(k-1)}) \end{aligned}$$

# Interpretation

- first iteration $(k = 1)$ is a proximal gradient step at $x^{(0)}$
- next iterations are proximal gradient steps at extrapolated points $y^{(k-1)}$

$$x^{(k)} = \mathrm{prox}_{t_k h}\left(y^{(k-1)} - t_k \nabla g(y^{(k-1)})\right)$$




sequence $x^{(k)}$ remains feasible (in $\mathbf{dom}\, h$); sequence $y^{(k)}$ not necessarily

# Convergence of FISTA

$$\text{minimize} \quad f(x) = g(x) + h(x)$$

**assumptions**

- optimal value $f^\star$ is finite and attained at $x^\star$ (not necessarily unique)
- $\mathbf{dom}\, g = \mathbf{R}^n$ and $\nabla g$ is Lipschitz continuous with constant $L > 0$
- $h$ is closed (implies $\mathrm{prox}_{th}(u)$ exists and is unique for all $u$)

**result:** with fixed step size $t_k = 1/L$

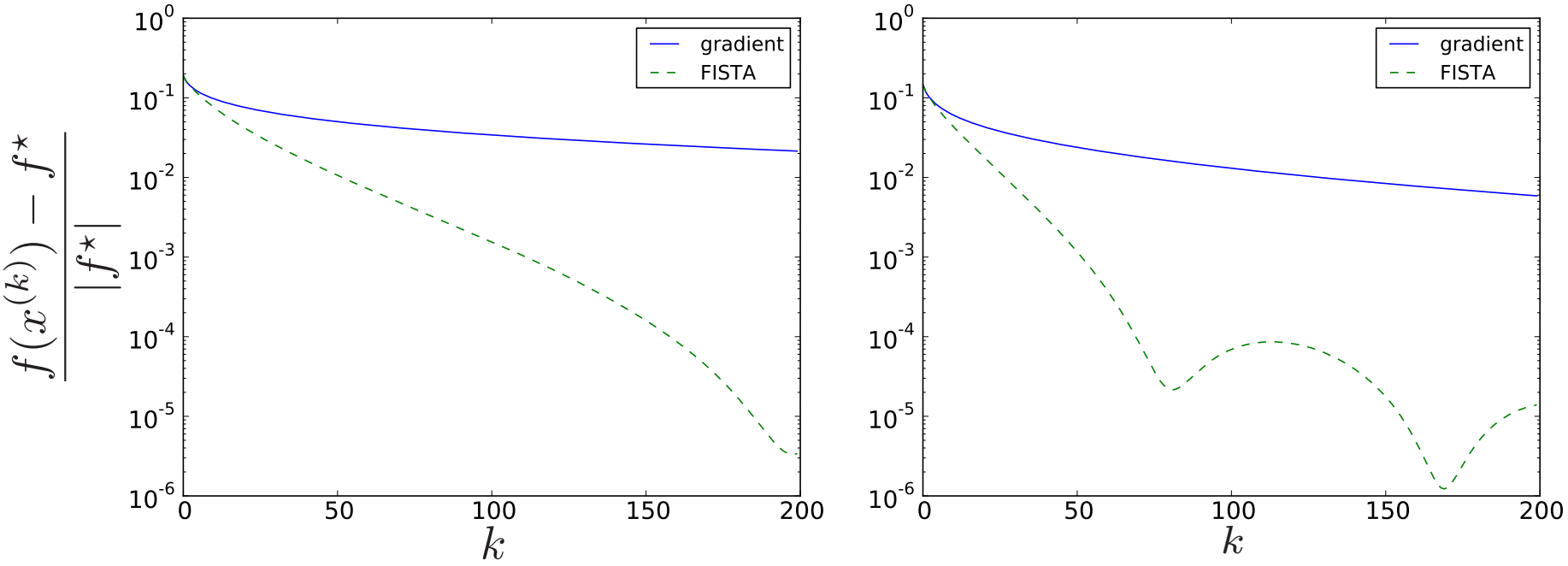
$$f(x^{(k)}) - f^\star \leq \frac{2L}{(k+1)^2}\|x^{(0)} - f^\star\|_2^2$$

- compare with $1/k$ convergence rate for gradient method
- can be extended to account for line searches

# Example

$$\text{minimize} \quad \log \sum_{i=1}^{m} \exp(a_i^T x + b_i)$$

randomly generated data with $m = 2000$, $n = 1000$, same fixed step size




FISTA is not a descent method

# Dual methods

- Lagrange duality
- dual decomposition
- dual proximal gradient method
- multiplier methods

# Dual function

**convex problem** (with linear constraints for simplicity)

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Gx \leq h \\ & Ax = b \end{array}$$

optimal value $p^\star$

**Lagrangian**

$$\begin{aligned} L(x, \lambda, \nu) &= f(x) + \lambda^T (Gx - h) + \nu^T (Ax - b) \\ &= f(x) + (G^T \lambda + A^T \nu)^T x - h^T \lambda - b^T \nu \end{aligned}$$

**dual function**

$$\begin{aligned} g(\lambda, \nu) &= \inf_x L(x, \lambda, \nu) \\ &= -f^*(-G^T \lambda - A^T \nu) - h^T \lambda - b^T \nu \end{aligned}$$

# Dual problem

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \geq 0 \end{array}$$

optimal value $d^\star$

a convex optimization problem in $\lambda$, $\nu$

**weak duality:** $p^\star \geq d^\star$, without exception

**strong duality:** $p^\star = d^\star$ if a constraint qualification holds

(for example, primal problem is feasible and $\mathbf{dom}\, f$ open)

# Least-norm solution of linear equations

$$\begin{array}{ll} \text{minimize} & f(x) = \|x\| \\ \text{subject to} & Ax = b \end{array}$$

recall that $f^*$ is indicator function of unit dual norm ball

**dual problem**

$$\text{maximize} \quad -b^T\nu - f^*(-A^T\nu) = \begin{cases} -b^T\nu & \|A^T\nu\|_* \leq 1 \\ -\infty & \text{otherwise} \end{cases}$$

**reformulated dual problem**

$$\begin{array}{ll} \text{minimize} & b^T z \\ \text{subjec to} & \|A^T z\|_* \leq 1 \end{array}$$

# Norm approximation

$$\text{minimize} \quad \|Ax - b\|$$

**reformulated problem**

$$\begin{array}{ll} \text{minimize} & \|y\| \\ \text{subject to} & y = Ax - b \end{array}$$

**dual function**

$$\begin{aligned} g(\nu) &= \inf_{x,y} \left( \|y\| + \nu^T y - \nu^T A x + b^T \nu \right) \\ &= \begin{cases} b^T \nu & A^T \nu = 0, \quad \|\nu\|_* \leq 1 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

**dual problem**

$$\begin{array}{ll} \text{minimize} & b^T z \\ \text{subject to} & A^T z = 0, \quad \|z\|_* \leq 1 \end{array}$$

# Karush-Kuhn-Tucker optimality conditions

if strong duality holds, then $x$, $\lambda$, $\nu$ are optimal if and only if

1. *primal feasibility:*

$$x \in \mathbf{dom}\, f, \qquad Gx \leq h, \qquad Ax = b$$

2. $\lambda \geq 0$

3. *complementary slackness:*

$$\lambda^T(h - Gx) = 0$$

4. $x$ minimizes $L(x, \lambda, \nu) = f(x) + \lambda^T(Gx - h) + \nu^T(Ax - b)$

   for differentiable $f$, condition 4 can be expressed as

$$\nabla f(x) + G^T\lambda + A^T\nu = 0$$

# Outline

- Lagrange dual
- **dual decomposition**
- dual proximal gradient method
- multiplier methods

# Dual methods

**primal problem**

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Gx \leq h \\ & Ax = b \end{array}$$

**dual problem**

$$\begin{array}{ll} \text{maximize} & -h^T\lambda - b^T\nu - f^*(-G^T\lambda - A^T\nu) \\ \text{subject to} & \lambda \geq 0 \end{array}$$

possible advantages of solving the dual when using first-order methods

- dual problem is unconstrained or has simple constraints
- dual problem can be decomposed into smaller problems

# (Sub-)gradients of conjugate function

$$f^*(y) = \sup_x \left(y^T x - f(x)\right)$$

- subgradient: $x$ is a subgradient at $y$ if it maximizes $y^T x - f(x)$
- if maximizing $x$ is unique, then $f^*$ is differentiable

  this is the case, for example, if $f$ is strictly convex

**strongly convex function:** $f$ is strongly convex with parameter $\mu$ if

$$f(x) - \frac{\mu}{2} x^T x \quad \text{is convex}$$

implies that $\nabla f^*(x)$ is Lipschitz continuous with parameter $1/\mu$

# Dual gradient method

primal problem with equality constraints and dual

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Ax = b \end{array}$$

**dual ascent:** use (sub-)gradient method to minimize

$$-g(\nu) = b^T\nu + f^*(-A^T\nu) = \sup_x \left((b - Ax)^T\nu - f(x)\right)$$

**algorithm**

$$\begin{aligned} x^+ &= \operatorname*{argmin}_x \left(f(x) + \nu^T Ax\right) \\ \nu^+ &= \nu + t(Ax^+ - b) \end{aligned}$$

of interest if calculation of $x^+$ is inexpensive (for example, separable)

# Dual decomposition

$$\begin{array}{ll} \text{minimize} & f_1(x_1) + f_2(x_2) \\ \text{subject to} & G_1 x_1 + G_2 x_2 \leq h \end{array}$$

objective is separable; constraint is *complicating* (or *coupling*) constraint

**dual problem** ('master' problem)

$$\begin{array}{ll} \text{maximize} & -h^T\lambda - f_1^*(-G_1^T\lambda) - f_2^*(-G_2^T\lambda) \\ \text{subject to} & \lambda \geq 0 \end{array}$$

can be solved by (sub-)gradient projection if $\lambda \geq 0$ is the only constraint

**subproblems:** for $j = 1, 2$, evaluate

$$f_j^*(-G_j^T\lambda) = -\inf_{x_j} \left( f_j(x_j) + \lambda^T G_j x_j \right)$$

maximizer $x_j$ gives subgradient $-G_j x_j$ of $f_j^*(-G_j^T\lambda)$ w.r.t. $\lambda$

## dual subgradient projection method

- solve two unconstrained (and independent) subproblems

$$x_j^+ = \operatorname*{argmin}_{x_j} \left( f_j(x_j) + \lambda^T G_j x_j \right), \quad j = 1, 2$$

- make projected subgradient update of $\lambda$

$$\lambda^+ = \left( \lambda + t(G_1 x_1^+ + G_2 x_2^+ - h) \right)_+$$

**interpretation:** price coordination between two units in a system

- constraints are limits on shared resources; $\lambda_i$ is price of resource $i$
- dual update $\lambda_i^+ = (\lambda_i - t s_i)_+$ depends on slacks $s = h - G_1 x_1 - G_2 x_2$
  - increases price $\lambda_i$ if resource is over-utilized ($s_i < 0$)
  - decreases price $\lambda_i$ if resource is under-utilized ($s_i > 0$)
  - never lets prices get negative

# Outline

- Lagrange dual
- dual decomposition
- **dual proximal gradient method**
- multiplier methods

# First-order dual methods

| | | | |
|---|---|---|---|
| minimize | $f(x)$ | maximize | $-f^*(-G^T\lambda - A^T\nu)$ |
| subject to | $Gx \geq h$ | subject to | $\lambda \geq 0$ |
| | $Ax = b$ | | |

**subgradient method:** slow, step size selection difficult

**gradient method:** faster, requires differentiable $f^*$

- in many applications $f^*$ is not differentiable, has a nontrivial domain
- $f^*$ can be smoothed by adding a small strongly convex term to $f$

**proximal gradient method (this section):** dual costs split in two terms

- first term is differentiable
- second term has an inexpensive prox-operator

# Composite structure in the dual

**primal problem with separable objective**

$$\begin{array}{ll} \text{minimize} & f(x) + h(y) \\ \text{subject to} & Ax + By = b \end{array}$$

**dual problem**

$$\text{maximize} \quad -f^*(A^T z) - h^*(B^T z) + b^T z$$

has the composite structure required for the proximal gradient method if

- $f$ is strongly convex; hence $\nabla f^*$ is Lipschitz continuous
- prox-operator of $h^*(B^T z)$ is cheap (closed form or efficient algorithm)

# Regularized norm approximation

$$\text{minimize} \quad f(x) + \|Ax - b\|$$

$f$ strongly convex with modulus $\mu$; $\|\cdot\|$ is any norm

**reformulated problem and dual**

$$\begin{array}{ll} \text{minimize} & f(x) + \|y\| \\ \text{subject to} & y = Ax - b \end{array} \qquad \begin{array}{ll} \text{maximize} & b^T z - f^*(A^T z) \\ \text{subject to} & \|z\|_* \leq 1 \end{array}$$

- gradient of dual cost is Lipschitz continuous with parameter $\|A\|_2^2/\mu$

$$\nabla f^*(A^T z) = \operatorname*{argmin}_x \left( f(x) - z^T A x \right)$$

- for most norms, projection on dual norm ball is inexpensive

**problem:** minimize $f(x) + \|Ax - b\|$

**dual gradient projection algorithm:** choose initial $z$ and repeat

$$\begin{aligned} \hat{x} &:= \operatorname*{argmin}_x \left(f(x) - z^T Ax\right) \\ z &:= P_C\left(z + t(b - A\hat{x})\right) \end{aligned}$$

- $P_C$ is projection on $C = \{y \mid \|y\|_* \leq 1\}$
- step size $t$ is constant or from backtracking line search
- can use accelerated gradient projection algorithm (FISTA) for $z$-update
- first step decouples if $f$ is separable

# Outline

- Lagrange dual
- dual decomposition
- dual proximal gradient method
- **multiplier methods**

# Moreau-Yosida regularization of the dual

a general technique for smoothing the dual of

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Ax = b \end{array}$$

- maximizing $g(\nu) = \inf_x \left(f(x) + \nu^T(Ax - b)\right)$ is equivalent to maximizing

$$g_t(\nu) = \sup_z \left(g(z) - \frac{1}{2t}\|\nu - z\|_2^2\right)$$

- from duality, $g_t(\nu) = \inf_x L_t(x, \nu)$ where

$$L_t(x, \nu) = f(x) + \nu^T(Ax - b) + (t/2)\|Ax - b\|_2^2$$

- $g_t$ is concave, differentiable with Lipschitz cont. gradient (constant $1/t$)

$$\nabla g_t(\nu) = A\hat{x} - b, \qquad \hat{x} = \operatorname*{argmin}_x L_t(x, \nu)$$

# Augmented Lagrangian method

**algorithm:** choose initial $\nu$ and repeat

$$\begin{aligned} x^+ &= \operatorname{argmin} L_t(x, \nu) \\ \nu^+ &= \nu + t(Ax^+ - b) \end{aligned}$$

- maximizes Moreau-Yosida regularization $g_t$ via gradient method
- $L_t$ is the augmented Lagrangian (Lagrangian plus quadratic penalty)

$$L_t(x, \nu) = f(x) + \nu^T(Ax - b) + \frac{t}{2}\|Ax - b\|_2^2$$

- method can be extended to problems with inequality constraints

# Dual decomposition

**convex problem with separable objective**

$$\begin{array}{ll} \text{minimize} & f(x) + h(y) \\ \text{subject to} & Ax + By = b \end{array}$$

**augmented Lagrangian**

$$L_t(x, y, \nu) = f(x) + h(y) + \nu^T(Ax + By - b) + \frac{t}{2}\|Ax + By - b\|_2^2$$

- difficulty: quadratic penalty destroys separability of Lagrangian
- solution: replace minimization over $(x, y)$ by alternating minimization

# Alternating direction method of multipliers

apply one cycle of alternating minimization steps to augmented Lagrangian

1. minimize augmented Lagrangian over $x$:

$$x^{(k)} = \operatorname*{argmin}_x L_t(x, y^{(k-1)}, \nu^{(k-1)})$$

2. minimize augmented Lagrangian over $y$:

$$y^{(k)} = \operatorname*{argmin}_y L_t(x^{(k)}, y, \nu^{(k-1)})$$

3. dual update:

$$\nu^{(k)} := \nu^{(k-1)} + t\left(Ax^{(k)} + By^{(k)} - b\right)$$

can be shown to converge under weak assumptions

# Sparse inverse covariance selection

$$\text{minimize} \quad \mathbf{tr}(CX) - \log\det X + \|X\|_1$$

variable $X \in \mathbf{S}^n$; $\|X\|_1$ is sum of absolute values of $X$

**reformulation**

$$\begin{array}{ll} \text{minimize} & \mathbf{tr}(CX) - \log\det X + \|Y\|_1 \\ \text{subject to} & X - Y = 0 \end{array}$$

**augmented Lagrangian**

$$\begin{aligned} & L_t(X, Y, Z) \\ & \quad = \quad \mathbf{tr}(CX) - \log\det X + \|Y\|_1 + \mathbf{tr}(Z(X - Y)) + \frac{t}{2}\|X - Y\|_F^2 \end{aligned}$$

**ADMM steps:** alternating minimization of augmented Lagrangian

$$\mathbf{tr}(CX) - \log\det X + \|Y\|_1 + \mathbf{tr}(Z(X-Y)) + \frac{t}{2}\|X-Y\|_F^2$$

- minimization over $X$:

$$\hat{X} = \operatorname*{argmin}_X \left( -\log\det X + \frac{t}{2}\|X - Y + \frac{1}{t}(C+Z)\|_F^2 \right)$$

  follows easily from eigenvalue decomposition of $Y - (1/t)(C+Z)$

- minimization over $Y$:

$$\hat{Y} = \operatorname*{argmin}_Y \left( \|Y\|_1 + \frac{t}{2}\|Y - \hat{X} - \frac{1}{t}Z\|_F^2 \right)$$

  apply element-wise soft-thresholding to $\hat{X} - (1/t)Z$

- dual update $Z := Z + t(\hat{X} - \hat{Y})$

cost per iteration dominated by cost of eigenvalue decomposition

# Sources and references

these lectures are based on the courses

- EE364A (S. Boyd, Stanford), EE236B (UCLA), *Convex Optimization*

  `www.stanford.edu/class/ee364a`
  `www.ee.ucla.edu/ee236b/`

- EE236C (UCLA) *Optimization Methods for Large-Scale Systems*

  `www.ee.ucla.edu/~vandenbe/ee236c`

- EE364B (S. Boyd, Stanford University) *Convex Optimization II*

  `www.stanford.edu/class/ee364b`

see the websites for expanded notes, references to literature and software